$SO(10)$ - Grand Unification and Fermion Masses

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DEUTSCHE ZUSAMMENFASSUNG

Nach dem neuesten Stand der Wissenschaft ist das Standard Model die erfolgreichste Theorie, die die Wechselwirkungen zwischen der Elementarteilchen genau beschreiben kann. Es umfasst alle fundamentalen Wechselwirkungen der Natur außer der Gravitation. Seine Vorhersagen wurden zu einer hohen Genauigkeit geprüft. Dennoch wird es nicht als die fundamentale Theorie der Eichwechselwirkungen betrachtet. Es hat zu viele unbestimmte Parameter. Es kann die Fermionenmassen nicht vorhersagen, und es gelingt ihm auch nicht, die geringen Neutrinomassen zu erklären, welche in der letzten Zeit durch Experimente bestätigt wurden. Es verfügt über keine Eichbosonen, die Nukleonzerfälle verursachen können, was für die Erklärung der Baryonenasymetrie des Universums erforderlich ist. Auch müssen CP-verletzende Phasen künstlich in die CKM oder MNS Matrizen eingeführt werden.

Die Nachteile des Standard Models kann man im Rahmen der großen vereinheitlichten Theorien beseitigen welche größere Freiheitsgrade besitzen. Große vereinheitlichte Theorien, welche nur eine Eichkopplung besitzen, basieren auf Eichgruppen, die die Standardmodeleichgruppe beinhalten. Es gibt eine limitierte Anzahl solcher Gruppen. $SO(10)$ ist eine voll symmetrische Eichgruppe, die über zwei Merkmale verfügt: Es vereinigt alle bekannten Wechselwirkungen unter einer Kopplung und klassifiziert alle bekannten Fermionen einer Familie in einem einzigen Spinor.

In dieser Arbeit untersuchen wir die große vereinheitlichte $SO(10)$ Theorie durch Anwendung verschiedener Matrizendarstellungen, welche die Struktur der $SO(10)$ klar zum Ausdruck bringen. Unsere Methode basiert auf zwei Schritten: Wir werden die expliziten Ausdrücke der Masseneigenwerte und Masseneigenzustände der physikalischen Eichbosonen von einer sogenannten quadrierten Massenmatrix ableiten, die über alle Informationen der Mischungsparametern zwischen Eichfeldern, und den Phasen die zur Quelle der CP-Verletzung dienen, verfügt. Mit Hilfe dieser Analyse werden wir die expliziten Ausdrücke der Wechselwirkungslagrangedichte der geladenen Ströme, ungeladenen Ströme und farbgeladenen Ströme der $SO(10)$ ableiten. Wir werden explizite Ausdrücke der Vektor- und Axialvektorkopplungen der ungeladenen zwei Ströme der $SO(10)$ darstellen. Wir werden die Baryonen-, Leptonen- und Baryonen- minus Leptonenzahl verletzenden Prozesse und deren CP verletzenden Phasen, die auf der $SO(10)$ beruhen, präsentieren.

Das Higgs Potenzial, das in den Higgs Mechanismus eingeführt wird, werden wir durch eine Bearbeitung der $SO(10)$ Higgsfelder im allgemeinsten Fall konstruieren, wobei wir insbesondere die ausdrückliche Matrizendarstellung der Higgsfelder veranschaulichen werden. Der potenzielle Teil der Higgs Lagrangedichte wird uns die Eigenschaften des Minimums des Vakuums, und der kinetische Teil wird uns die quadrierte Massenmatrix der Eichbosonen durch eine spontane Symmetriebrechung liefern. Die Higgsfelder werden an den Fermionen mit Hilfe einer demokratischen Yukawakopplung gekoppelt. Dadurch werden wir explizite Ausdrücke der Fermionenmassen der dritten Generation erhalten, einschließlich der Majorana und Dirac Massen der Neutrinos. Wir werden eine Flavour-Eigenbasis für die Neutrinos einführen und die Masseneigenwerte und die Masseneigenzustände der Neutrinos finden. Explizite Ausdrücke für die CP-Verletzung im Neutrinosektor werden angegeben.

In dem zweiten Schritt dieser Arbeit, werden wir sämtliche oben genannten Größen auswerten. Wir werden unsere Auswertungen mit bekannten Größen aus dem Standard Model wie den W und Z Bosonenmassen, der Vektor- und Axialvektorkopplung des ungeladenen Stromes und den Fermionenmassen der dritten Generation vergleichen. Zusätlich werden wir Größen wie Massen neuer Eichbosonen, Vektor- und Axialvektorkopplungen eines neuen ungeladenen Stromes, leichte Massen der linkshändigen und schwere Massen der rechtshändigen Neutrinos, Werte verschiedener Mischungsparametern und CP verletzende Phasen usw. die jeweils nicht aus dem Standard Model bekannt sind, präsentieren.

Die zu obigen Auswertungen benötigten Eingabewerte werden hauptsächlich durch zwei Quellen erworben: Zuerst werden wir die Vakuumerwartungswerte und die Eichkopplungen der $SO(10)$ Wechselwirkungen im Rahmen der Vereinigung der Kopplungen durch Untersuchung der $SO(10)$ Massenskalen so gut wie möglich bestimmen. Ergänzend, werden wir die Vakuumerwartungswerte und deren Phasen durch Justierung an die genau gemessenen Massen der bekannten Eichbosonen und Fermionen, die jeweils unter der Fermiskala liegen, bestimmen. Es wird uns gelingen, über 67 Parameter mit Hilfe von 7 Erwartungswerten, 5 Winkeln, einer Eichkopplung und einer Yukawakopplung vorherzusagen.

ABSTRACT

In the state of the art the Standard Model is the best gauge theory describing interactions among elementary particles. It comprises all of the fundamental interactions in nature except gravitation. Its predictions have been experimentally tested to a high level of accuracy. However, it is not considered to be the fundamental theory of gauge interactions. It contains a lot of arbitrary parameters. It can not predict the fermion masses and fails to explain the smallness of neutrino masses which have been observed by recent experiments. It contains no gauge bosons that can mediate nucleon decays via baryon and lepton number violating process, which are needed to explain the baryon asymmetry in our universe. Furthermore, CP violation has to be introduced into the CKM and MNS matrices by hand.

The shortcomings of the Standard Model can be solved in the framework of grand unified gauge theories (GUTs) which have greater degrees of freedom. GUT's which have truly one coupling constant are based on gauge groups that contain the Standard Model as a subgroup. There are a limited number of such gauge groups. $SO(10)$ is a fully symmetric gauge group that has two outstanding features: It unifies all the known gauge interactions under a single coupling strength and classifies all the known fermions of a family under a single spinor.

In this work, we will study $SO(10)$ grand unification in its full extent by using different explicit matrix representations which exhibit the structure of $SO(10)$ in a very transparent way. Our approach consists mainly of two stages: We will derive the explicit expressions of the mass-eigenvalues and mass-eigenstates of the physical gauge bosons from a mass squared-matrix that contains all the information about the mixing parameters among the gauge fields and the phases which are sources for CP violation. In the light of this analysis, we will derive the explicit expressions for the interaction Lagrangians of the charged currents, the neutral currents and the charged and colored currents in $SO(10)$. We will present explicit expressions of the vector and axial-vector couplings of the two neutral currents in $SO(10)$. We will show how the baryon, lepton and baryon minus lepton number violating processes and their explicit CP violating phases are accommodated in the $SO(10)$ theory.

The Higgs potential that we use to implement in the Higgs mechanism will be constructed in a most general fashion through a careful study of the Higgs fields of $SO(10)$, where we give special emphasis on illustrating the explicit matrix representation of these Higgs fields. The potential part of the Higgs Lagrangian will give us the properties of the minimum of the vacuum, and the kinetic part will give us the mass-squared matrix of the gauge bosons via spontaneous symmetry breakdown. The same Higgs multiplets will be coupled to fermions through a democratic Yukawa matrix. Thereby, we will derive explicit expressions for the fermion masses of the third family including Majorana and Dirac masses for neutrinos. We will introduce a flavor-eigenbasis for neutrinos and find the mass-eigenstates and mass-eigenvalues of the neutrinos. Explicit expressions for CP violation in the neutrino sector will be obtained.

In the second stage of our work, we will evaluate all the above mentioned quantities. We will compare our results with those of the Standard Model like the W and Z masses and the vector and axial-vector coupling of the NC current and the fermion masses of the third family. In addition, we will present the values of the physical quantities that are not present in the Standard Model like the masses of new gauge bosons, the vector and axial-vector couplings of a new NC current, the masses of a light left-handed and a heavier right-neutrino, the values of various mixing parameters and CP phases etc.

The input values required for these evaluations will be acquired mainly from two sources: First, we will determine the vacuum expectation values and the coupling strengths of gauge interactions given by the $SO(10)$ theory in so far as possible through studying the mass scales in $SO(10)$ in the framework of coupling unification. Complementarily, we will determine the vacuum expectation values and their phases by adjusting them to the masses of the known gauge bosons and fermions below the Fermi scale which are accurately measured and known. We will be able to predict more than 67 parameters with an input of 7 vacuum expectation values, 5 angles, 1 gauge coupling and 1 Yukawa coupling.

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1. INTRODUCTION

Unification is presumably one of the most central claims of particle physics [1]. It furnishes the basis of the great achievement of the Glashow-Weinberg-Salam theory of electroweak gauge interactions commonly known as the electroweak theory [2][3][4][5][6]. The electroweak gauge theory is based on the $SU(2)_L \times U(1)_Y$ direct product gauge group and allows us to study the electromagnetic and weak interactions which have been regarded for long as separate interactions, in a single framework. Strictly speaking their theory is not a true unification. Since every gauge group requires its own gauge coupling, we are still dealing with two interactions having no common source. A pleasing situation could have been achieved if they were able to relate the separate gauge couplings in the electroweak theory through simple relations that follow from the properties of the involved gauge groups. Unfortunately the two gauge couplings g and g' assigned to the $SU(2)_L \times U(1)_Y$ direct product respectively are just related over a mixing angle [7]. We have

$$\frac{g'}{g} = \tan\theta_W \tag{1.1}$$

where θ_W is a free parameter of the theory [8][9]. In this brief introduction, we do not attempt to give a concise historical evolution of particle physics but rather aim at bringing important facts together that underlie some of the major steps leading to the idea of grand unified theories [10]. Let us continue with another widely accepted claim of particle physics: In the state of the art all interactions among elementary particles are described by gauge theories [11][12][13][14]. Indeed the electromagnetic theory has been the forerunner of gauge theories [15]. The strong interaction has been successfully described as a gauge interaction as well, particularly based on the $SU(3)_c$ gauge group [16][17][18]. The strong interaction together with the electroweak interaction give the so called standard model based on the $SU(3)_c \times SU(2)_L \times U(1)_Y$ direct product gauge group [14]. To date there is no known discrepancy between the standard model and experiments.

Due to the two claims highlighted above and the failure of the electroweak model being a true unification it turned out to be most natural to consider a gauge theory possessing a single gauge coupling and containing the standard model gauge group as a subgroup, to be a candidate grand unifying theory of all fundamental interactions as reviewed in ref. [19]. The pioneering grand unified gauge theory satisfying the above requirements has been proposed by S. L. Glashow and H. Georgi to be the $SU(5)$ theory [20]. This theory has the same rank as the standard model but contains more degrees of freedom. It predicts the existence of additional gauge bosons which reside in the coset of the respective gauge groups. Actually this is always the faith of grand unified theories that every gauge group with a rank higher or equal to 4 will bring up new interactions. Consequently we are compelled to seek for new physics beyond the standard model in one way or the other. In the $SU(5)$ theory, these hypothetical gauge bosons do mediate the proton to decay into a positron and a neutral pion [21][22]. We have the reaction

$$p \to e^+ + \pi^0 \tag{1.2}$$

Unfortunately the $SU(5)$ theory predicts the proton to decay in $2 \times 10^{29\pm1.7}$ years [23] which is faster than the recently measured lower bound [24] and therefore the theory is in serious trouble [25]. The proton life time is sensitive to the gauge boson masses that mediate the decay process [26][27][28][29]. In general, gauge bosons mediating nucleon decays get masses at the order of the so-called grand unification mass scale at which the spontaneous breakdown of the symmetry occurs. One can also implement a chain of spontaneous symmetry breaking which is usually the case. The grand unification mass scale is the energy scale at which coupling unification is achieved: The coupling strengths of separate interactions associated with the various subgroups of the single gauge group are subject to renormalization as we evolve them towards higher energies [30][31]. The behavior of the couplings strengths of Abelian and non-Abelian gauge theories at short distances are different. The latter type gauge theories for which the coupling strength at short distances decreases are referred to as asymptotically free [32][33][34], whereas in the former type theories, the coupling strength at short distances increases. The energy scale at which the strength of the couplings become equal determines the grand unification mass scale and should lie considerably high to avoid any unwanted effect like proton decay [35][36]. This requirement sounds tricky but the non-observation of the proton decay implies us that grand unified theories should deal with extremely high energies. The grand unification mass scale of the $SU(5)$ theory lies roughly at $3.1 \times 10^{14\pm0.3}$ GeV which is relatively low [23].

From the other side, on rather aesthetical grounds one might find it unpleasant to observe that the fermions of a single family can not be assigned to a single fermion multiplet in the $SU(5)$ theory which is another shortcoming of the theory [23]. This means that it does not satisfactorily predict the family structure of fermions. It also excludes the existence of a right handed neutrino which might seem as an advantage in the first place because this particle fails to exist, but recent findings suggesting that neutrinos have tiny masses have turned the existence of a right-handed neutrino into an attractive and interesting problem [37][38][39]. Finally we find it appropriate to mention that the $SU(5)$ theory does not explain why nature favors $V-A$ currents over $V+A$. This is commonly known as the left-right asymmetry observed in nature and requires in our opinion further explanation by any candidate grand unified theory [40][41][42][43][44].

Of course the quest for grand unification does not end here. Another candidate gauge group for grand unification has been proposed by H. Fritzsch and P. Minkowski to be the $SO(10)$ theory [45][46]. This theory has rank 5 and provides more degrees of freedom which makes it phenomenologically very attractive. It provides a rich framework and addresses many problems remnant of the electroweak theory and even cosmology [47]. Some immediate features of the theory will be instantly summarized in the following:

One of the most striking feature special to $SO(10)$ is that it accommodates all the observed fermions of a family including the missing right handed neutrino within a single fermion representation. Through the eigenvalue operators of $SO(10)$ one can fix various known charges of elementary particles. The spinorial representation of $SO(10)$ and the related eigenvalue operators which make this possible will be given in § 2 and § 3 respectively. Through this feature, it successfully classifies our known spectrum of elementary particles [46]. But unfortunately it fails to give any hint why families repeat.

Furthermore it suggests an initially left-right symmetric universe prior to any spontaneous symmetry breakdown. The left-right symmetry imposed by the $SO(10)$ theory becomes obvious when its structure is studied. This will mainly be done in § 3. This feature serves us a framework to study why physics close to the Fermi scale, best described by the electroweak theory favors left-handed currents over right handed ones.

Another interesting feature is that it allows us to endow neutrinos both with Majorana and Dirac masses [10]. This feature can give rise to the existence of very massive right-handed neutrinos and almost massless left-handed neutrinos. The formal framework for studying the asymmetric behavior of neutrino masses is commonly called the see-saw mechanism and is naturally suggested by the theory. Thereby the $SO(10)$ theory indirectly accounts for the non-observation of the right-handed neutrino below the experimentally accessible Fermi scale. In the SO(10) theory the masses of the leptons and quarks will be achieved through the Yukawa couplings in conjunction with the Higgs mechanism [48][49][50][51]. The Yukawa sector of $SO(10)$ will be mainly studied in § 12. But before that a detailed knowledge of the $SO(10)$ Higgs sector is essentially required [52][53]. The various Higgs fields that are physically most relevant will be studied in § 6, 7, 8, 9 and 10. As will be shown later in § 10, the observed left-right asymmetry of nature can closely be linked to the fact that left-handed neutrinos are almost massless and right-handed neutrinos are so heavy that they may only be produced in extremely energetic processes.

We will also show in § 11.2 that the minimization of the Higgs potential can describe a left-right asymmetric vacuum under a specific condition. This condition will later help us to estimate the values of various quantities in $SO(10)$. We are unfortunately faced with the fact that our ignorance about the Higgs couplings in the Higgs potential makes it impossible to evaluate the vevs from the minimum of the Higgs potential despite of the fact that we can solve the minimum for each of the vevs separately. Therefore we have to find the values of the vevs in that we make use of the standard model and the electroweak theory parameters like the W and Z masses as well as the strong and the electromagnetic interaction couplings etc. These procedure is mainly studied in § 14 and § 15.

Furthermore in the $SO(10)$ theory, CP violation can be induced by assigning certain Higgs fields that transform under the $SO(10)$ gauge group with non-trivial complex phases. The Higgs sector of $SO(10)$ is extremely rich and offers great amount of freedom to study CP violation. The complex phases which induce CP violation are collectively introduced in § 11 and will be evaluated in § 15.

An additional feature of the $SO(10)$ theory is that it accommodates new gauge bosons apart from those we know from the $SU(5)$ theory which can mediate baryon and separately lepton number violating processes. These gauge bosons however conserve local $B-L$ number at the vertices. Indeed the $B-L$ number appears as the charge of a local $U(1)_{B-L}$ gauge symmetry which naturally embeds itself into the $SO(10)$ gauge group when a certain isomorphism between unitary and orthogonal groups are considered. This isomorphism and additional features of orthogonal groups are studied in § 2. The properties of these $B-L$ carrying gauge bosons and the various interactions mediated by them will be studied in great detail in § 3 and § 5 respectively.

A novel feature of $SO(10)$ is that it also allows local $B-L$ $violating$ gauge interactions to occur. These mechanisms rest on the transitions of quarks into leptons where the quarks and leptons sit in the same multiplet leading to the so called lepton-quark unification [54][55]. Lepton-quark unification is based on the $SU(4)_c$ gauge group which also embeds itself into $SO(10)$ via an isomorphism [56]. These aspects are also studied in § 3 and § 5.

The $SO(10)$ theory predicts a relatively high grand unification mass scale that prohibits the undesired fast decay of the proton [36]. Some estimates of the grand unification mass scales and the intermediate mass scales which are decisive on the masses of heavy gauge bosons will be given in § 14 where we mainly study the coupling unification in the realm of $SO(10)$.

The $SO(10)$ theory contains the electroweak theory as a sub-theory. Therefore it should be possible to recover various predictions and expressions of the physical observables of the electroweak theory [57][36]. From the other side, e.g. the expressions for the masses of the gauge bosons in the $SO(10)$ theory, will not be as simple as those in the electroweak theory. We expect that the former expressions reduce to the latter ones *if* we switch off some of the relevant parameters. These parameters are mostly the vacuum expectation values pertaining to the intermediate mass scales governing the overall Higgs mechanism. The determination of the expressions for the gauge boson masses requires a detailed study of the Higgs sector present in the $SO(10)$ theory. We will give a special emphasis on finding exact expressions for gauge bosons masses. In this respect, a mass-squared matrix of the gauge bosons will be given in § 11.3. As will be seen later, the Higgs scalars can give rise to certain mixing among the gauge fields as they become massive. Consequently we will have to reexpress the various interaction Lagrangians in terms of the physical gauge fields and the physical currents which can be classified into 3 types. These will be the charged currents, the neutral currents and the charged currents which simultaneously carry color. These currents will be studied in § 13. Such an analysis also allows us to see how the CP violating phases come into the Lagrangians.

Another feature of $SO(10)$ grand unification is that it has the necessary ingredient to produce a net excess of matter over anti-matter. This will be studied in § 13.

The overall symmetry breaking pattern and various vacuum expectation values and their phases as well as the mixing among gauge fields and the resulting mass eigenstates and the expressions for the mixing parameters and more will be all studied in § 11. Finally numerical estimates of the quark and lepton masses and the gauge boson masses as well as their mixing parameters and their CP phases will be presented in § 15.

The above mentioned features and few more related with the $SO(10)$ theory will be elaborated in great detail through out this work. In the remaining part of this brief introduction, we find it appropriate to deal with some of the fundamental aspects of gauge theories which are believed to underlie all elementary particle physics. In particular we will continue our excursion by briefly introducing the formal basis of the $SO(10)$ gauge theory [58] [47].

The gauge principle is understood as the invariance of a Lagrangian with respect to certain types of transformations which enable interactions to occur. Essentially it is demanded that these transformations are local, i.e., the rotation specifying parameters, say ω, are co-ordinate dependent. In other words, we are insisting that a global invariance holds locally as well. Such theories are known as local gauge theories [59]. The set of matrices which induce transformations are chosen to form a group which is called the gauge group and the rotation specifying parameters give rise to the existence of a new vector field called the gauge field. This vector field naturally requires its own free Lagrangian which will be introduced in the next lines. If the vector field should not spoil the local invariance of the Lagrangian, we have to demand the gauge fields to be initially massless. The massless gauge fields acquire mass through the Higgs mechanism which will be implemented in our model in § 11. In order to give a self contained and short transition to the gauge theoretical formulation of the $SO(10)$ theory, we highlight some basics steps in the procedure. These steps constitute the main approach, no matter what particular gauge group one deals with. Let Σ be the representation matrices for the fermions. The spinorial transformation of a spinor can be formally stated as

$$\Psi_a \to \left(e^{-i\,\Sigma\cdot\omega}\right)_{ab} \Psi_b = U_{ab}\ \Psi_b \tag{1.3}$$

The indices ab indicate entries of the matrix representation of the exponent. A more conventional approach is to use the notation on the right hand side in the above expression where U is a unitary matrix and denotes the matrix representation of the exponential term and Ψ_a is a spinor accommodating the fermions of a complete family which is achievable in $SO(10)$. The entries and the size of the spinor will be studied primarily in § 4. For the moment the explicit form of the representation matrices Σ_{ab} of $SO(10)$ are not interesting to us. They will be explicitly introduced in § 2. As it is well known from local gauge theories, the transformation of kinetic terms which involve partial derivatives generate non-invariant terms. Let us consider the transformation of the partial derivative of the spinor. We have

$$\partial_\mu\ \Psi \to U\ \partial_\mu \Psi + (\partial_\mu\ U)\ \ \Psi \tag{1.4}$$

where the last term in the expression becomes an undesired term. Since we insist on imposing local gauge invariance, it would be trivial to set U to a constant value. The well known remedy is rather to replace the partial derivative with a so-called covariant derivative. In this way the Lagrangian can be made locally gauge invariant. This is the general technique adopted in local gauge theories. The covariant derivative is defined as

$$D_\mu = \partial_\mu + i\ g\ W_\mu \cdot \Sigma \tag{1.5}$$

Here W_{ab} are real valued $N(N-1)/2$ vector gauge fields with $a, b = 1, \ldots, N; N = 10$ and Σ_{ab} are the antisymmetric representation matrices and g is the coupling strength. The above inner product implies a sum over the group indices ab. Note that if g vanishes not only the interactions but the terms defining local gauge invariance also disappear in the theory. Furthermore D_μ is a matrix in the space of group indices. The second term appearing in the covariant derivative is usually known as the gauge term matrix. We will give special attention to the gauge term matrix and present its physical content in § 4 using different representations of Σ. The physical gauge fields of the theory are always complex valued linear combinations of W_{ab}. The number of independent physical gauge fields is determined by the degrees of freedom possessed by the unification gauge group. For $SO(10)$, we have 45 gauge fields. The expressions of the physical gauge fields in terms of the W_{ab}'s will be studied mainly in § 3. Under a gauge transformation, the vector fields W_μ should transforms in such a way that the term $\partial_\mu\, U$ disappears. This is equivalent to expect the transformation of the covariant derivative of the spinor to be of the following form

$$D_\mu \Psi \rightarrow U\; D_\mu \Psi \tag{1.6}$$

This condition yields the desired transformation rule of the gauge term $W_\mu \cdot \Sigma$. We have

$$W_\mu \cdot \Sigma \rightarrow {W_\mu}' \cdot \Sigma \equiv U\,(W_\mu \cdot \Sigma)\,U^{-1} + \frac{i}{g}\,(\partial_\mu U)\,U^{-1} \tag{1.7}$$

A further step is to find the transformation of W_μ. This can be derived from the above equation. U and Σ do not commute in general. For the sake of simplicity, we can expand U around the identity by neglecting second order terms in the expansion. We have

$$U \cong\ I\ -\ i\,g\,\omega\cdot\Sigma\,, \quad U^{-1} \cong\ I\ +\ i\,g\,\omega\cdot\Sigma, \quad \partial_\mu\,U \cong -\,i\,g\,\partial_\mu\omega\cdot\Sigma \tag{1.8}$$

The transformation rule of the gauge field W_μ can be approximately obtained by substituting the above expansions into eq. (1.7). By neglecting higher order terms during the intermediate steps, we obtain

$${W_\mu}' \cdot \Sigma \equiv W_\mu \cdot \Sigma + i\,g\,[\,\omega\cdot\Sigma\,,\ W_\mu\cdot\Sigma\,] +\ \partial_\mu\omega\cdot\Sigma \tag{1.9}$$

The above commutator can be handled by using the commutation and anti-commutation relations among the Σ matrices and their generating Γ basis respectively which will be introduced in § 2. The commutator simplifies and we get the infinitesimal transformation rule of the gauge fields. We have

$$W_{ab\ \ \mu} \rightarrow W_{ab\ \ \mu}\ +\ g\,\omega_{ac}\,W_{cb\ \mu}\ +\ g\,\omega_{bc}\,W_{ca\ \mu} + \partial_\mu\omega_{ab} \tag{1.10}$$

Trough comparing the last two equations, it is seen that we have projected out the Σ's in the latter one. Finally we may add to the Lagrangian a gauge invariant kinetic energy term for each of the ${W_\mu}^{ab}$ fields. The gauge invariant kinetic energy term is build from the field strength ${F_{\mu\nu}}^{ab}$ which is defined as

$$\begin{aligned} {F_{\mu\nu}}^{ab} &= \partial_\mu {W_\nu}^{ab} - \partial_\nu {W_\mu}^{ab} - g\left({W_\mu}^{ac}{W_\nu}^{cb} - {W_\nu}^{ac}{W_\mu}^{cb}\right) \\ {F_{\mu\nu}}^{ab} &\rightarrow {F_{\mu\nu}}^{ab} + g\left(\omega^{ac}F^{cb}_{\mu\nu} - \omega^{ac}{F_{\nu\mu}}^{cb}\right) \end{aligned} \tag{1.11}$$

In the second line above, we have shown how the field strength transforms under a local gauge transformation. The final Lagrangian will be composed of the Lagrangian of the massless spinor field Ψ_a and the Lagrangian of the massless vector fields ${W_{\mu\nu}}^{ab}$. We have

$$\mathcal{L} = \bar\Psi\; i\,\gamma^\mu\; D_\mu\;\Psi - \frac{1}{4}\,F^{\mu\nu\ ab}{F_{\mu\nu}}^{ab} =\ \underbrace{\bar\Psi\; i\,\gamma^\mu\;\partial_\mu\Psi}_{\text{Kinetic energy of }\Psi}\ -\ \underbrace{g\;\bar\Psi\;(\,W_\mu\cdot\Sigma\,)\;\Psi}_{\text{Interaction}} -\ \underbrace{\frac{1}{4}\,F^{\mu\nu}\cdot F_{\mu\nu}}_{\text{Kinetic energy of W's}} \tag{1.12}$$

where the first term produces the Dirac equation and the second term is the kinetic energy of the gauge fields. The second term on the right hand side above contains all the interaction terms contained in the $SO(10)$ theory. These are the fermion currents coupling to the various gauge fields through the coupling strength g. These will be presented in § 5.

We also need to define a suitable Higgs Lagrangian that we can implement into the Higgs mechanism of our $SO(10)$ model. This will be done in § 11.

2. SOME FEATURES OF ORTHOGONAL GROUPS

2.1 Real Representation of $SO(N)$

Since the standard model is based on unitary groups, we find it appropriate to recall few elementary features of orthogonal groups before we start exploring the physical ingredient of a physically viable $SO(10)$ model.

In m dimensions one can define $m(m-1)/2$ linearly independent and antisymmetric matrices to form a basis such that any real antisymmetric $m \times m$ matrix, say Σ, can be expanded in terms of this basis with $m(m-1)/2$ coefficients of ω_{ab} where $(a, b = 1, 2, ..., m)$. Orthogonal rotations in m dimension can be obtained by exponentiating such antisymmetric real $m \times m$ matrices. The coefficients ω_{ab} in the former expansion will determine finite angles of rotations. Rotations in m dimension can be expressed as

$$R_m = e^{-i\ \Sigma_{ab} \cdot \omega_{ab}} \tag{2.1}$$

where Σ_{ab} are the basis and ω_{ab} are rotation specifying real valued parameters which appeared as expansion coefficients. If the above rotation, acting on vectors, induce preserved length than this rotation will satisfy the condition $R_m {R_m}^T = 1$. A suitable generating expression for the basis Σ_{ab} where $(a, b, c, d = 1, 2, ..., m)$ can be stated as

$$(\Sigma_{ab})_{cd} = \delta_{ac}\delta_{bd} - \delta_{bc}\delta_{ad} \tag{2.2}$$

where the indices cd are showing the entries of the matrix Σ_{ab} and the indices ab are the labels of the element in the basis [56]. Note that the number of degrees of freedom, i.e. the independent ways of possible rotations in three dimensions is three, that‘s why commonly $SO(3)$ generators or the $SO(3)$ basis is labelled with a single index running from 1 to 3. But for higher dimension this is no good convention any more. It is seen that in this basis for $a \neq b$, Σ_{ab} will have $zeros$ everywhere except at positions (a, b) and (b, a). These entries are occupied by $+1$ and -1 respectively and additionally we have $\Sigma_{ab} = -\Sigma_{ba}$. The Lie algebra of Σ_{ab} is given through

$$[\Sigma_{ab}, \Sigma_{cd}] = \delta_{ad}\Sigma_{bc} + \delta_{bc}\Sigma_{ad} - \delta_{ac}\Sigma_{bd} - \delta_{bd}\Sigma_{ac} \tag{2.3}$$

This expression can be constructed using the representation $[\,\partial/\partial X_j, X_i] = \delta_{ij}$, which yields the angular momentum generators used in quantum mechanics for $(i, j = 1, 2, 3)$. We have

$$\Sigma_{ab} = X_a \frac{\partial}{\partial X_b} - \frac{\partial}{\partial X_b} X_a \qquad a, b = 1, 2, ..., m \tag{2.4}$$

It is seen from the Lie algebra of $SO(m)$ that (a) two generators will commute when they do not have any common index and (b) a non-zero commutation arises when they have $just\ one$ common index and no more. It is useful to note that any non-zero commutation yields on the right hand side a single generator although the right hand side of the expression is crowded in terms. The mutually commuting generators can be found using the property stated in (a); They are $\Sigma_{12}, \Sigma_{34}, \Sigma_{56}, ...$ These generators form an $Abelian$ subgroup i.e., the $Cartan$ Subalgebra of $SO(m)$. The rank of the algebra is equal to the number of mutually commuting generators.

2.2 Spinorial Representation of $SO(N)$

The spinorial representation of orthogonal groups appears in its simplest and clearest form in the $SO(3)$ case [56]. Since $SU(2)$ is locally isomorphic to $SO(3)$, a spinorial representation of $SO(3)$ and hence spinorial finite transformations can be constructed using the basis of the $SU(2)$ algebra through straightforward exponentiation

$$R_3 = e^{-i\ \sigma_{ab} \cdot \omega_{ab}} \tag{2.5}$$

σ_{ab} are the Pauli matrices and ω_{ab} are rotation specifying parameters. Note that in this context the Pauli matrices are labelled with two indices. The isomorphism between $SU(2)$ and $SO(3)$ is equal to the fact that the Pauli matrices σ_a also satisfy the Lie algebra of $SO(3)$ if they are expressed as

$$\sigma_{ab} = \frac{i}{4}[\sigma_a, \sigma_b] \qquad a, b = 1, 2, 3 \tag{2.6}$$

$$[\sigma_{ab}, \sigma_{cd}] = i\,(\,\delta_{ad}\,\sigma_{bc} + \delta_{bc}\,\sigma_{ad} - \delta_{ac}\,\sigma_{bd} - \delta_{bd}\,\sigma_{ac}\,) \tag{2.7}$$

It is seen that the Lie algebra above is similar to the real valued one of $SO(3)$ given in eq. (2.3) except for the factor $-i$ appearing on the right hand side. This can be met by putting a $-i$ in front of the right hand side in eq. (2.2) so that the algebra is complexified [56]. But in fact this modification yields only imaginary valued antisymmetric matrices. To see the nature of the spinorial transformation of $SO(3)$, it is good to look at the following example [60]. It also serves as key point in generalizing the $SO(3)$ spinorial representations to higher dimensions. Consider a complex valued 2×2 matrix M. The expansion of M in terms of σ_a yields ; $M = x\sigma_1 + y\sigma_2 + z\sigma_3$. A unitary transformation acting on M such that $M' = U^\dagger M U$ induces orthogonal rotation on the coordinates (i.e., the coefficients in the expansion) x , y and z or equivalently on $\sigma_1, \sigma_2, \sigma_3$. We have

$$\begin{array}{lll} x' & = \;\; x\,\cos(2\beta) + y\,\sin(2\beta) & \sigma_1{}' \;\; = \;\; \sigma_1\,\cos(2\beta) + \sigma_1\,\sin(2\beta) \\ y' & = -y\,\sin(2\beta) + x\,\cos(2\beta) & \sigma_2{}' \;\; = -\sigma_1\,\sin(2\beta) + \sigma_2\,\cos(2\beta) \\ z' & = \;\; z & \sigma_3{}' \;\; = \;\; \sigma_3 \end{array} \tag{2.8}$$

where U is chosen to be $diag\ (e^{i\beta}, e^{-i\beta})$. It is seen that the unitary transformation U induces a double valued orthogonal transformation due to the argument 2β in the sines and the cosines. Furthermore the following quantity is left invariant

$$x^2 + y^2 + z^2 = (x\,\sigma_1 + y\,\sigma_2 + z\,\sigma_2)^2 \tag{2.9}$$

The generalization from 3 dimensions to higher dimensions is straightforward and can be achieved through introducing m traceless unitary Γ matrices such that the length of an m component vector is left invariant:

$${x_1}^2 + {x_2}^2 + ... + {x_m}^2 = (x_1\Gamma_1 + x_2\Gamma_2 + ... + x_m\Gamma_m)^2 \tag{2.10}$$

The requirement that the sum of the squares is equal to the square of the sums will restrict the gamma matrices to satisfy the Clifford Algebra which is obviously fulfilled by the Pauli matrices for the $SO(3)$ case as well [58]. We have

$$\{\Gamma_a\,\Gamma_b \;+\; \Gamma_b\,\Gamma_a\} = 2\,\delta_{ab}\,\mathbb{1} \tag{2.11}$$

Here δ_{ab} carries Euclidean signature and $\mathbb{1}$ is a unit matrix with appropriate size. The Σ_{ab} basis of the spinorial $SO(m)$ representation will be generated by these Γ matrices. These matrices will also satisfy the $Lie\ Algebra\ of$ $SO(m)$ with the property $\Sigma_{ab} = -\Sigma_{ba}$. We have

$$\Sigma_{ab} = \frac{i}{4}[\Gamma_a, \Gamma_b] \tag{2.12}$$

$$[\Sigma_{ab}, \Sigma_{cd}] = \;i\,(\,\delta_{ad}\,\Sigma_{bc} + \delta_{bc}\,\Sigma_{ad} - \delta_{ac}\,\Sigma_{bd} - \delta_{bd}\,\Sigma_{ac}\,) \tag{2.13}$$

where $a, b = (1, 2, ..., m)$. Using the Σ_{ab} basis, the spinorial transformation is finally obtained and is similar to the one in eq. (2.5). We have

$$R_m = e^{-i\;\Sigma_{ab}\cdot\omega_{ab}} \tag{2.14}$$

Note that $R_m {R_m}^\dagger = 1$ and $\Sigma_{ab} = {\Sigma_{ab}}^\dagger$. This can be verified by looking at infinitesimal transformation. i.e, Σ_{ab} are either real valued symmetric matrices or imaginary valued antisymmetric matrices.

2.3 An Explicit Basis for the Spinorial Representation

There is no *general* way to write down a basis that can produce the generators of the spinorial representation of $SO(N)$. A conventionally useful way is to iterate the Pauli matrices using the tensor product while keeping at each step of iteration the *Clifford Algebra* satisfied [61]. In this technique each tensor product increases the rank of the subsequently resulting representation by *one*. The size of the matrix representation doubles itself as well. We should have in mind that the rank of $SO(2m)$ and $SO(2m+1)$ are equal. A basis produced through m iterations will be a spinorial basis for each of one them. But $SO(2m+1)$ requires one more Γ matrix as an element in the basis. The Spinorial representation of $SO(2m)$ is reducible and $SO(2m+1)$ is irreducible. The irreducible spinor of $SO(2m+1)$ will transform under a matrix representation with size 2^m , and is self conjugate and real. The spinor of $SO(2m)$ is for all m reducible into two pieces, each with dimension 2^{m-1}. These two pieces are (i) real and self conjugate when m is even and (ii) complex when m is odd. The complex spinors in (ii) are conjugate to each other and are called *chiral or Weyl spinors*. It should be noted that although the spinors are reducible for $SO(2m)$, the representation with the dimension 2^m may or may not be always block wise reducible into a size of 2^{m-1}. i.e., the reducible spinors may transform under a matrix representation with size 2^m. A chirality operator can be defined to illuminate this fact:

$$\Gamma_{five} = (-i)^{m/2}\Gamma_1\Gamma_2\cdots\Gamma_m \tag{2.15}$$

This will be a matrix of dimension 2^m and in some cases is block wise reducible depending on the Γ matrices. In this case the chirality operator assumes the form

$$\Gamma_{five} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{2.16}$$

The entry 1 is a $2^{m-1} \times 2^{m-1}$ identity matrix. The reduced spinors and the representation under which they transform is obtained through Γ_{five}. We have

$$\Sigma^{\pm} = \frac{1}{2}\left(1 \pm \Gamma_{five}\right)\, \Sigma \quad , \qquad \Psi^{\pm} = \frac{1}{2}\left(1 \pm \Gamma_{five}\right)\, \Psi \tag{2.17}$$

Here the $\pm$ signs in $\Sigma^{\pm}_{ab}$ simply indicate that they only transform the spinor $\Psi^{\pm}$ with the respective sign. $\Psi^{\pm}$ correspond to the chiral components of the spinor Ψ. If Γ_{five} derived from a particular basis assumes the above form then Ψ and Σ might be taken as 16 and 16×16 dimensional objects respectively and can be suitably expressed as

$$\Psi = \begin{pmatrix} \Psi^+ \\ \Psi^- \end{pmatrix}, \;\; \Sigma = \begin{pmatrix} \Sigma^+ & 0 \\ 0 & \Sigma^- \end{pmatrix} \tag{2.18}$$

If a block wise reducible representation can not be achieved then the chiral entries of the spinor are distributed over 32 dimensions and the generators are in size 32×32. Then we write $\Psi = \Psi^+ + \Psi^-$ and $\Sigma = \Sigma^+ + \Sigma^-$ respectively. In the remaining part we introduce 3 differently obtained bases A,B,C.

2.3.1 Basis A

The following one is a good example for a block wise reducible representation [61]. It is obtained by successively multiplying the second and third lines by 1 from the right where 1 is a 2×2 unit matrix. The multiplication is understood to be a tensor product. The identity $(A_1 \times B_1) \cdot (A_2 \times B_2) = (A_1 \cdot A_2) \times (B_1 \cdot B_2)$ can be used to verify that the Clifford algebra given in eq. (2.11) is satisfied at each step of iteration. We have

$$\begin{array}{ccccc}
\Gamma_1 & = & \sigma_1 \times \sigma_1 \times \sigma_1 \times & ..... & \times\, \sigma_1 \times \sigma_1 \times \sigma_1 \\
\Gamma_2 & = & \sigma_1 \times \sigma_1 \times \sigma_1 \times & ..... & \times\, \sigma_1 \times \sigma_1 \times \sigma_2 \\
\Gamma_3 & = & \sigma_1 \times \sigma_1 \times \sigma_1 \times & ..... & \times\, \sigma_1 \times \sigma_1 \times \sigma_3 \\
\Gamma_4 & = & \sigma_1 \times \sigma_1 \times \sigma_1 \times & ..... & \times\, \sigma_1 \times \sigma_2 \times\, 1 \\
\Gamma_5 & = & \sigma_1 \times \sigma_1 \times \sigma_1 \times & ..... & \times\, \sigma_1 \times \sigma_3 \times\, 1 \\
\Gamma_6 & = & \sigma_1 \times \sigma_1 \times \sigma_1 \times & ..... & \times\, \sigma_2 \times\, 1\, \times\, 1 \\
\Gamma_7 & = & \sigma_1 \times \sigma_1 \times \sigma_1 \times & ..... & \times\, \sigma_3 \times\, 1\, \times\, 1 \\
\vdots & = & \vdots & & \vdots \\
\Gamma_{2m-2} & = & \sigma_1 \times \sigma_2 \times 1 \times & ..... & \times 1\, \times 1\, \times\, 1 \\
\Gamma_{2m-1} & = & \sigma_1 \times \sigma_3 \times 1 \times & ..... & \times 1\, \times 1\, \times\, 1 \\
\Gamma_{2m} & = & \sigma_2\, \times 1\, \times 1 \times & ..... & \times 1\, \times 1\, \times\, 1 \\
\hline
\Gamma_{2m+1} & = & \sigma_3\, \times 1\, \times 1 \times & ..... & \times 1\, \times 1\, \times\, 1
\end{array} \tag{2.19}$$

Here σ_a are the usual Pauli matrices where $a = 1, 2, 3$. The first $2m$ Γ matrices will produce the spinorial representation of $SO(2m)$. If the last Γ_{2m+1} is included, we obtain the spinorial representation of $SO(2m+1)$. It is seen that $\Gamma_{five} = \sigma_3 \times 1 \times 1 \times 1 \times \cdots \times 1$. The diagonal generators of $SO(10)$ for iterations up to $m = 5$ are implicitly

$$\begin{array}{ccc}
\Sigma_{12} & = & 1\, \times\, 1\, \times\, 1\, \times\, 1\, \times \sigma_3 \\
\Sigma_{34} & = & 1\, \times\, 1\, \times\, 1\, \times \sigma_3 \times \sigma_3 \\
\Sigma_{56} & = & 1\, \times\, 1\, \times \sigma_3\, \times \sigma_3 \times \sigma_3 \\
\Sigma_{78} & = & 1\, \times \sigma_3\, \times \sigma_3\, \times \sigma_3 \times \sigma_3 \\
\Sigma_{910} & = & \sigma_3\, \times \sigma_3\, \times \sigma_3\, \times \sigma_3 \times \sigma_3
\end{array} \tag{2.20}$$

Looking at any of the Σ's as well as the non-diagonals, it is seen that in the first tensor product they evolve along the diagonals such that all off diagonal entries are *zero*. As Γ_{five} is in the form given in eq. (2.16) we may conclude that the representation transforming the chiral spinor have size 2^{m-1}.i.e, for $SO(10)$ they are of size 16.

2.3.2 Basis B

This basis is in particular not block wise reducible. It is often used in the literature [52]. The iteratively obtained basis is generated from the following set of equations:

$$\begin{aligned}
\Gamma_{2k} &= \underbrace{1 \times 1 \times ..... \times 1}_{k-1\ times} \times \sigma_2 \times \underbrace{\sigma_3 \times \sigma_3 \times ..... \times \sigma_3}_{m-k\ times} \\
\Gamma_{2k-1} &= \underbrace{1 \times 1 \times ..... \times 1}_{k-1\ times} \times \sigma_1 \times \underbrace{\sigma_3 \times \sigma_3 \times ..... \times \sigma_3}_{m-k\ times}
\end{aligned} \tag{2.21}$$

Here k starts with 1. There are $k-1$ tensor products of 1 multiplying σ_2 from the left and $m-k$ tensor products of σ_3 multiplying σ_2 from the left. The convention for the Pauli matrices are again as usual. The diagonal generators following from eq. (2.12) appear to be $\Sigma_{12}, \Sigma_{34}, \cdots$ and are

$$\Sigma_{2k,2k-1} = -\underbrace{1 \times 1 \times ..... \times 1 \times 1}_{k-1\ times} \times \sigma_3 \times \underbrace{1 \times 1 \times 1 \times ..... \times 1}_{m-k\ times} \tag{2.22}$$

There are $k-1$ tensor products of 1 acting on σ_3 from the left and $m-k$ tensor products of 1 acting on from the right. Note that the tensor product is associative but not commutative. $\Gamma_{five} = \sigma_3 \times \sigma_3 \times \sigma_3 \cdots$ It is seen that during the iteration, Γ_{2k} for $k=1$ evolves not along the diagonal as Γ_{2k-1}. This spoils somehow the possibility of having 16 dimensional matrices. It should be from now on expected that many gauge fields will lie in off diagonal blocks of size 16 in the adjoint representation. We have

$$\begin{aligned}
\Gamma_1 &= \sigma_1 \times \sigma_3 \times \sigma_3 \times \sigma_3 \times \sigma_3 & \Gamma_2 &= \sigma_2 \times \sigma_3 \times \sigma_3 \times \sigma_3 \times \sigma_3 \\
\Gamma_3 &= 1 \times \sigma_1 \times \sigma_3 \times \sigma_3 \times \sigma_3 & \Gamma_4 &= 1 \times \sigma_2 \times \sigma_3 \times \sigma_3 \times \sigma_3 \\
\Gamma_5 &= 1 \times 1 \times \sigma_1 \times \sigma_3 \times \sigma_3 & \Gamma_6 &= 1 \times 1 \times \sigma_2 \times \sigma_3 \times \sigma_3 \\
\Gamma_7 &= 1 \times 1 \times 1 \times \sigma_1 \times \sigma_3 & \Gamma_8 &= 1 \times 1 \times 1 \times \sigma_2 \times \sigma_3 \\
\Gamma_9 &= 1 \times 1 \times 1 \times 1 \times \sigma_1 & \Gamma_{10} &= 1 \times 1 \times 1 \times 1 \times \sigma_2
\end{aligned} \tag{2.23}$$

It is also possible to do it the other way around. Indeed, we prefer to re-label the Γ basis; Γ_1, Γ_2, Γ_3, Γ_4 will be labelled as Γ_7, Γ_8, Γ_9, Γ_{10} respectively and $\Gamma_5, \ldots, \Gamma_{10}$ will be called $\Gamma_1, \ldots, \Gamma_6$.The new assignment is

$$\begin{aligned}
\Gamma_7 &= \sigma_1 \times \sigma_3 \times \sigma_3 \times \sigma_3 \times \sigma_3 & \Gamma_8 &= \sigma_2 \times \sigma_3 \times \sigma_3 \times \sigma_3 \times \sigma_3 \\
\Gamma_9 &= 1 \times \sigma_1 \times \sigma_3 \times \sigma_3 \times \sigma_3 & \Gamma_{10} &= 1 \times \sigma_2 \times \sigma_3 \times \sigma_3 \times \sigma_3 \\
\Gamma_1 &= 1 \times 1 \times \sigma_1 \times \sigma_3 \times \sigma_3 & \Gamma_2 &= 1 \times 1 \times \sigma_2 \times \sigma_3 \times \sigma_3 \\
\Gamma_3 &= 1 \times 1 \times 1 \times \sigma_1 \times \sigma_3 & \Gamma_4 &= 1 \times 1 \times 1 \times \sigma_2 \times \sigma_3 \\
\Gamma_5 &= 1 \times 1 \times 1 \times 1 \times \sigma_1 & \Gamma_6 &= 1 \times 1 \times 1 \times 1 \times \sigma_2
\end{aligned} \tag{2.24}$$

Both assignments satisfy the generalized form of the $SO(10)$ basis that will be introduced in § 3. There is no strict requirement behind this re-labelling. The difference arises mainly in the appearance of the gauge term matrix which amounts to a redistribution of the physical fields. The latter is more convenient and will be used in § 4. We have

$$\begin{aligned}
\Sigma_{78} &= \sigma_3 \times 1 \times 1 \times 1 \times 1 \\
\Sigma_{910} &= 1 \times \sigma_3 \times 1 \times 1 \times 1 \\
\Sigma_{12} &= 1 \times 1 \times \sigma_3 \times 1 \times 1 \\
\Sigma_{34} &= 1 \times 1 \times 1 \times \sigma_3 \times 1 \\
\Sigma_{56} &= 1 \times 1 \times 1 \times 1 \times \sigma_3
\end{aligned} \tag{2.25}$$

2.3.3 Basis C

A third basis that we introduce has a non-iterative structure and is rather hand made for $SO(10)$ [62]. Its obvious advantage becomes clear when the gauge field structure is constructed and various embedding are done. This basis is block wise reducible and contains the $SU(4)$ and $SU(3)$ subgroups in its fundamental representation, i.e. a repetitive $SU(4)$ and $SU(3)$ structure along the diagonal, which is not achieved in the previous introduced ones. The Γ basis is defined as

$$\begin{aligned}
\Gamma_1 &= \sigma_1 \times \sigma_1 \times 1 \times 1 \times \sigma_2 & \Gamma_6 &= \sigma_1 \times \sigma_2 \times 1 \times \sigma_1 \times \sigma_2 \\
\Gamma_2 &= \sigma_1 \times \sigma_2 \times 1 \times \sigma_3 \times \sigma_2 & \Gamma_7 &= \sigma_1 \times \sigma_3 \times \sigma_1 \times 1 \times 1 \\
\Gamma_3 &= \sigma_1 \times \sigma_1 \times 1 \times \sigma_2 \times \sigma_3 & \Gamma_8 &= \sigma_1 \times \sigma_3 \times \sigma_2 \times 1 \times 1 \\
\Gamma_4 &= \sigma_1 \times \sigma_2 \times 1 \times \sigma_2 \times 1 & \Gamma_9 &= \sigma_1 \times \sigma_3 \times \sigma_3 \times 1 \times 1 \\
\Gamma_5 &= \sigma_1 \times \sigma_1 \times 1 \times \sigma_2 \times \sigma_1 & \Gamma_{10} &= \sigma_2 \times 1 \times 1 \times 1 \times 1
\end{aligned} \tag{2.26}$$

Here Γ_{five} is as in eq. (2.16) and given as $\Gamma_{five} = \sigma_3 \times 1 \times 1 \times 1 \times 1$. The Abelian subgroup of diagonal generators is given by:

$$\begin{array}{lcl} \Sigma_{12} & = & 1 \ \times \sigma_3 \times 1 \ \times \sigma_3 \times 1 \\ \Sigma_{34} & = & 1 \ \times \sigma_3 \times 1 \ \times 1 \ \times \sigma_3 \\ \Sigma_{56} & = & 1 \ \times \sigma_3 \times 1 \ \times \sigma_3 \times \sigma_3 \\ \Sigma_{78} & = & 1 \ \times 1 \ \times \sigma_3 \times 1 \ \times 1 \\ \Sigma_{910} & = & \sigma_3 \times \sigma_3 \times \sigma_3 \times 1 \ \times 1 \end{array} \tag{2.27}$$

2.4 The Maximal Subgroup $SO(6) \times SO(4)$ and $U(5)$

A maximal subgroup of a group G is by definition a subgroup of G which has not a lower rank than the group G itself. The generators of this maximal subgroup can be selected out of the 45 generators of $SO(10)$ through the following choice.

$$\begin{array}{llcll} (a) & \Sigma_{SO(6)} & \equiv & \{\forall \ \Sigma_{ij} \mid & i,j = 1,2,3,4,5,6\} \\ (b) & \Sigma_{SO(4)} & \equiv & \{\forall \ \Sigma_{ij} \mid & i,j = 7,8,9,10\} \end{array} \tag{2.28}$$

The groups entering the direct product or the $generators$ of each group in (a) and (b) do mutually commute. The above maximal subgroup can also be obtained from the Dynkin diagram of $SO(10)$ [56]. This can be formally stated as

$$[\Sigma_{SO(6)}, \Sigma_{SO(4)}] \equiv \ 0 \tag{2.29}$$

and can be verified using the Lie algebra given in eq. (2.13). It should be noted that it is not always the case to assign the first six and the last four indices to $\Sigma_{SO(6)}$ and $\Sigma_{SO(4)}$ respectively. The choice can be inverted, which induces a redistribution of the fields within various matrix representations of the Higgs fields and gauge fields. However the physical content will be left unaltered. The rank of $SO(10)$ is 5, and it is seen that $SO(6) \times SO(4)$ has the same rank amounting to a maximal subgroup.

Another maximal subgroup of $SO(10)$ is the $U(5)$. One can equivalently transform a 10-component real vector as a five component complex vector [47]. The content of $U(5)$ in terms of the $SO(10)$ generators will postponed to a further section, because the correspondence between the generators of $U(5)$ and $SO(10)$ follow from an embedding procedure.

2.5 Some Isomorphisms in $SO(10)$

As we deal with spinors so the unitary representations will be required. This means that we will not use the Σ's of $SO(6)$ as single objects but consider certain combinations which produce the isomorphically equivalent unitary group $SU(4)$. Note that they have the same number of generators. We have

$$\begin{array}{ll} U_1 = (\Sigma_{45} + \Sigma_{36})/2 & U_9 = (\Sigma_{23} + \Sigma_{14})/2 \\ U_2 = (\Sigma_{53} + \Sigma_{46})/2 & U_{10} = (\Sigma_{31} + \Sigma_{24})/2 \\ U_3 = (\Sigma_{65} + \Sigma_{43})/2 & U_{11} = (\Sigma_{25} + \Sigma_{61})/2 \\ U_4 = (\Sigma_{52} + \Sigma_{61})/2 & U_{12} = (\Sigma_{51} + \Sigma_{62})/2 \\ U_5 = (\Sigma_{15} + \Sigma_{62})/2 & U_{13} = (\Sigma_{45} + \Sigma_{63})/2 \\ U_6 = (\Sigma_{23} + \Sigma_{41})/2 & U_{14} = (\Sigma_{53} + \Sigma_{64})/2 \\ U_7 = (\Sigma_{31} + \Sigma_{42})/2 & U_{15} = (\Sigma_{21} + \Sigma_{43} + \Sigma_{56})/(\sqrt{6}) \\ U_8 = (2\Sigma_{21} + \Sigma_{34} + \Sigma_{65})/(2\sqrt{3}) & \end{array} \tag{2.30}$$

These generators U_k satisfy the Lie algebra of $SU(4)$, where the right hand side is subject to the Lie algebra of $SO(6)$. We have

$$[U_k, U_\mu] = i f_{k\mu\nu} \, U_\nu \tag{2.31}$$

There is summation over ν, where $k, \mu, \nu = (1, \ldots, 15)$. The structure constants are summarized in Table (2.1) [63]. Another possible isomorphism in $SO(10)$ applies to the $SO(4)$ part of the maximal subgroup. The generators of the $SO(4)$ group can be organized in the following form

$$\begin{array}{ll} L_1 = (\Sigma_{79} + \Sigma_{108})/2 & R_1 = (\Sigma_{79} + \Sigma_{810})/2 \\ L_2 = (\Sigma_{98} + \Sigma_{107})/2 & R_2 = (\Sigma_{98} + \Sigma_{710})/2 \\ L_3 = (\Sigma_{87} + \Sigma_{109})/2 & R_3 = (\Sigma_{87} + \Sigma_{910})/2 \end{array} \tag{2.32}$$

Thereby $SO(4)$ becomes isomorphic to $SU(2)_L \times SU(2)_R$ [56]. The above embeddings fulfill the following Lie algebras:

$$[U_k, L_j] \equiv [U_k, R_j] \equiv 0, \quad [R_i, R_j] = i\epsilon_{ijk} R_k, \quad [L_i, L_j] = i\epsilon_{ijk} L_k, \quad [L_j, R_i] \equiv 0 \tag{2.33}$$

The subscripts L and R differentiate the $SU(2)$ groups in the $SU(2)_L \times SU(2)_R$ product and have no physical meaning unless one considers a definite assignment of the elementary particles to spinors under which they transform.

k	μ	ν	$f_{k\mu\nu}$	k	μ	ν	$f_{k\mu\nu}$	k	μ	ν	$f_{k\mu\nu}$	k	μ	ν	$f_{k\mu\nu}$
1	2	3	1	4	9	14	$\frac{1}{2}$	6	7	8	$\frac{\sqrt{3}}{2}$	8	9	10	$\frac{1}{2\sqrt{3}}$
1	4	7	$\frac{1}{2}$	4	10	13	$-\frac{1}{2}$	1	9	12	$\frac{1}{2}$	8	11	12	$\frac{1}{2\sqrt{3}}$
1	5	6	$-\frac{1}{2}$	5	9	13	$\frac{1}{2}$	1	10	11	$-\frac{1}{2}$	8	13	14	$-\frac{1}{\sqrt{3}}$
2	4	6	$\frac{1}{2}$	5	10	14	$\frac{1}{2}$	2	9	11	$\frac{1}{2}$	9	10	15	$\sqrt{\frac{2}{3}}$
2	5	7	$\frac{1}{2}$	6	11	14	$\frac{1}{2}$	2	10	12	$\frac{1}{2}$	11	12	15	$\sqrt{\frac{2}{3}}$
3	4	5	$\frac{1}{2}$	6	12	13	$-\frac{1}{2}$	3	9	10	$\frac{1}{2}$	13	14	15	$\sqrt{\frac{2}{3}}$
3	6	7	$-\frac{1}{2}$	7	11	13	$\frac{1}{2}$	3	11	12	$-\frac{1}{2}$				
4	5	8	$\frac{\sqrt{3}}{2}$	7	12	14	$\frac{1}{2}$								

Tab. 2.1: Non-zero structure constants $f_{k\mu\nu}$ of $SU(4)$, the $f_{k\mu\nu}$ is antisymmetric under permutation of any two indices. These structure constants match exactly with those of the Fundamental representation constructed as in the Gell-Man way.

3. THE STRUCTURE OF $SO(10)$

3.1 The Fields and Generators

In this section, we shortly introduce the general form of all the fields and generators, namely the 45 that will be used in the remaining parts of this work. The analysis is based on the $SO(6) \times SO(4)$ maximal subgroup [46]. The general form we introduce is not unique but convenient. This work contains three separate $SO(10)$ models which are physically equivalent and are derived from different bases found in literature. Therefore a common prescription might be necessary, e.g. U_{G_1} in any of the three models is set equal to $(U_1 + i\ U_2)/2$. The gauge fields related to this generator carry the same quantum numbers in any of the models (bases). On the other side the Σ_{ab} content of, for example, U_1 is the same in all models, but the entries of Σ_{ab} is representation dependent. We mainly adapted this procedure to reduce the size of the text, and to develop a unique perspective through out this work. An important property of this general structure is that it applies to all models in our work and obeys the same Lie algebra of the fundamental representation given in eqs. (2.31) and (2.33). Our definitions are as follows: we define for $SU(4)$, 15 real valued V_i fields and 15 U_i generators. All generators are 32×32 in size. They act on a 32 component spinor which contains the right and left handed 16 fermions. The assignment of the fermions are representation dependent and will be given for each model separately. The 8 gauge fields G_i of the subgroup $SU(3)_c$ and their 8 raising and lowering generators U_i are :

$$\begin{aligned}
G_1 &= \bar{G}_4 = (V_1 + iV_2)/\sqrt{2} & U_{G_1} &= U^\dagger_{G_4} = (U_1 - iU_2)/2 \\
G_2 &= \bar{G}_5 = (V_4 + iV_5)/\sqrt{2} & U_{G_2} &= U^\dagger_{G_5} = (U_4 - iU_5)/2 \\
G_3 &= \bar{G}_6 = (V_6 + iV_7)/\sqrt{2} & U_{G_3} &= U^\dagger_{G_6} = (U_6 - iU_7)/2 \\
G_7 &= \bar{G}_7 = (V_3\sqrt{3} + V_8)/2 & U_{G_7} &= \ (U_3/\sqrt{3} + U_8)/\sqrt{2} \\
G_8 &= \bar{G}_8 = (-V_3\sqrt{3} + V_8)/2 & U_{G_8} &= \ (-U_3/\sqrt{3} + U_8)/\sqrt{2}
\end{aligned} \tag{3.1}$$

6 of the remaining fields are grouped into two parts which are conjugated to each other. These are the lepto-quark fields.

$$\begin{aligned}
X_1 &= \bar{X}_4 = (V_9 + iV_{10})/\sqrt{2} & U_{X_1} &= U^\dagger_{X_4} = (U_9 - iU_{10})/2 \\
X_2 &= \bar{X}_5 = (V_{11} + iV_{12})/\sqrt{2} & U_{X_2} &= U^\dagger_{X_5} = (U_{11} - iU_{12})/2 \\
X_3 &= \bar{X}_6 = (V_{13} + iV_{14})/\sqrt{2} & U_{X_3} &= U^\dagger_{X_6} = (U_{13} - iU_{14})/2
\end{aligned} \tag{3.2}$$

We denote the $B - L$ eigenvalue generator with U_{B-L}. The corresponding B-L gauge field will be denoted with X_{B-L}. We have

$$B - L = U_{B-L} = 2\sqrt{2/3}U_{15}, \qquad X_{B-L} = V_{15} \tag{3.3}$$

The gauge fields and generators of $SO(4) \equiv SU(2)_L \times SU(2)_R$ in any model are defined as

$$\begin{aligned}
W^+_L &= (W^1_L + iW^2_L)/\sqrt{2} & L_+ &= (L_1 - iL_2)/2 \\
W^0_L &= W^3_L & L_0 &= L_3 \\
W^-_L &= (W^1_L - iW^2_L)/\sqrt{2} & L_- &= (L_1 + iL_2)/2
\end{aligned} \tag{3.4}$$

Here W^i_L and W^i_R for $i = (1, 2, 3)$ are real valued scalar fields. L_i are R_i are the $SU(2)_L$ and $SU(2)_R$ generators respectively. The physical gauge fields $W^{\pm,0}_L$ and $W^{\pm,0}_R$ are defined as

$$\begin{aligned}
W^+_R &= (W^1_R + iW^2_R)/\sqrt{2} & R_+ &= (R_1 - iR_2)/2 \\
W^0_R &= W^3_R & R_0 &= R_3 \\
W^-_R &= (W^1_R - iW^2_R)/\sqrt{2} & R_- &= (R_1 + iR_2)/2
\end{aligned} \tag{3.5}$$

The elements of the two groups are by definition always commuting. The above fields make the $SO(6) \times SO(4)$ part of $SO(10)$. There are 24 more gauge fields which belong to $SO(10)$ and lie outside the maximal subgroup $SO(6) \times SO(4)$. We denote their generators with S_i which are as before 32×32 matrices. The raising and lowering generators of these 24 gauge fields can be constructed from the $S_{1,..,24}$ generators. These raising and lowering generators will be denoted with D_i. The first 12 can be grouped into two mutually hermitian conjugate parts. We have

$$\begin{aligned} D_k &= (S_k - iS_{k+6})/2 \\ D_{k+6} &= (S_k + iS_{k+6})/2 \end{aligned} \tag{3.6}$$

where $k = 1, ..., 6$. $D_1, \ldots, D_6$ and $D_7, \ldots, D_{12}$ are coupling to new bosons which we denote with $A_1, ..., A_6$ and their conjugates with $\bar{A}_1, ..., \bar{A}_6$ respectively. The next 12 D generators are grouped similarly as

$$\begin{aligned} D_{k+12} &= (S_{k+12} - iS_{k+18})/2 \\ D_{k+18} &= (S_{k+12} + iS_{k+18})/2 \end{aligned} \tag{3.7}$$

where $k = 1, ..., 6$. These $D_{13}, \ldots, D_{18}$ and $D_{19}, \ldots, D_{24}$ couple to bosons which we denote with $Y_1, \ldots, Y_6$ and their conjugates with $\bar{Y}_1, \ldots, \bar{Y}_6$ respectively. These gauge bosons historically emerged first in the $SU(5)$ context of grand unification. The former ones are special to SO_{10}. To make things look more tidy, we adapt here a further convention: The first three bosons in $A_1, \ldots, A_6$ will be denoted with A_r, A_g, A_b and the last three with A'_r, A'_g, A'_b. The indices show SU_3 color. Also the first three bosons in $Y_1, \ldots, Y_6$ will be denoted with Y_r, Y_g, Y_b and the last three with Y'_r, Y'_g, Y'_b. This convention is not arbitrary and will be shown to hold once the charges of these gauge fields are recovered. The set of D generators should be handled correspondingly. We let $\{D_1, D_2, D_3\} \equiv D_{A_\alpha}$, $\{D_3, D_4, D_5\} \equiv D_{A'_\alpha}$, $\{D_{13}, D_{14}, D_{15}\} \equiv D_{Y_\alpha}$ and finally $\{D_{16}, D_{17}, D_{18}\} \equiv D_{Y'_\alpha}$, where α denotes color i.e., r, g, b in each triplet. Furthermore we introduce 24 real scalar fields $\mathbb{V}_i$ that make the A_α , A'_α and Y_α , Y'_α gauge fields. We have

$$\begin{aligned}
A_r = A_1 = (\mathbb{V}_1 + i\mathbb{V}_7)/\sqrt{2} \qquad & Y_r = Y_1 = (\mathbb{V}_{13} + i\mathbb{V}_{19})/\sqrt{2} \\
A_g = A_2 = (\mathbb{V}_2 + i\mathbb{V}_8)/\sqrt{2} \qquad & Y_g = Y_2 = (\mathbb{V}_{14} + i\mathbb{V}_{20})/\sqrt{2} \\
A_b = A_3 = (\mathbb{V}_3 + i\mathbb{V}_9)/\sqrt{2} \qquad & Y_b = Y_3 = (\mathbb{V}_{15} + i\mathbb{V}_{21})/\sqrt{2} \\
\\
A'_r = A_4 = (\mathbb{V}_4 + i\mathbb{V}_{10})/\sqrt{2} \qquad & Y'_r = Y_4 = (\mathbb{V}_{16} + i\mathbb{V}_{22})/\sqrt{2} \\
A'_r = A_5 = (\mathbb{V}_5 + i\mathbb{V}_{11})/\sqrt{2} \qquad & Y'_g = Y_5 = (\mathbb{V}_{17} + i\mathbb{V}_{23})/\sqrt{2} \\
A'_r = A_6 = (\mathbb{V}_6 + i\mathbb{V}_{12})/\sqrt{2} \qquad & Y'_b = Y_6 = (\mathbb{V}_{18} + i\mathbb{V}_{24})/\sqrt{2}
\end{aligned} \tag{3.8}$$

The generators U_i for $i = (1, \ldots, 15)$ of $SU(4)$ and L_i for $i = 1, 2, 3$ of $SU(2)_L$ and R_i for $(i = 1, 2, 3)$ of $SU(2)_R$ in terms of Σ's are collectively defined as

$$\begin{array}{llll}
U_1 & = (\Sigma_{45} + \Sigma_{36})/2 & & \\
U_2 & = (\Sigma_{53} + \Sigma_{46})/2 & L_1 & = (\Sigma_{79} + \Sigma_{108})/2 \\
U_3 & = (\Sigma_{65} + \Sigma_{43})/2 & L_2 & = (\Sigma_{98} + \Sigma_{107})/2 \\
U_4 & = (\Sigma_{52} + \Sigma_{61})/2 & L_3 & = (\Sigma_{87} + \Sigma_{109})/2 \\
U_5 & = (\Sigma_{51} + \Sigma_{62})/2 & & \\
U_6 & = (\Sigma_{23} + \Sigma_{41})/2 & R_1 & = (\Sigma_{79} + \Sigma_{810})/2 \\
U_7 & = (\Sigma_{31} + \Sigma_{24})/2 & R_2 & = (\Sigma_{98} + \Sigma_{710})/2 \\
U_8 & = (2\Sigma_{21} + \Sigma_{34} + \Sigma_{65})/(2\sqrt{3}) & R_3 & = (\Sigma_{87} + \Sigma_{910})/2 \\
 & & & \\
U_9 & = (\Sigma_{23} + \Sigma_{14})/2 & & \\
U_{10} & = (\Sigma_{31} - \Sigma_{42})/2 & & \\
U_{11} & = (\Sigma_{25} + \Sigma_{61})/2 & & \\
U_{12} & = (\Sigma_{51} + \Sigma_{62})/2 & & \\
U_{13} & = (\Sigma_{45} + \Sigma_{63})/2 & & \\
U_{14} & = (\Sigma_{53} + \Sigma_{64})/2 & & \\
 & & & \\
U_{15} & = (\Sigma_{21} + \Sigma_{43} - \Sigma_{65})/(\sqrt{6}) & &
\end{array} \tag{3.9}$$

The normalization of the basis is such that $Tr(U_iU_j) = Tr(L_iL_j) = Tr(R_iR_j) = 4\delta_{ij}$. The Lie algebra of the above set of U_i basis is given in Table (2.1). The generators satisfy ;$[U_k, U_\mu] = if_{k\mu\nu}U_\nu$ where there is a summation over ν for $k, \mu, \nu = 1, \ldots, 15$. The S_i basis related with the gauge fields lying outside the maximal subgroup for $i = 1, \ldots 24$ will be defined in terms of Σ_{ab} as

$$\begin{array}{llllll} S_1 & = & (\Sigma_{75} + \Sigma_{68})/2 & S_{13} & = & (\Sigma_{95} + \Sigma_{610})/2 \\ S_2 & = & (\Sigma_{37} + \Sigma_{48})/2 & S_{14} & = & (\Sigma_{39} + \Sigma_{410})/2 \\ S_3 & = & (\Sigma_{71} + \Sigma_{82})/2 & S_{15} & = & (\Sigma_{91} + \Sigma_{102})/2 \\ S_4 & = & (\Sigma_{59} + \Sigma_{610})/2 & S_{16} & = & (\Sigma_{75} + \Sigma_{86})/2 \\ S_5 & = & (\Sigma_{93} + \Sigma_{410})/2 & S_{17} & = & (\Sigma_{37} + \Sigma_{84})/2 \\ S_6 & = & (\Sigma_{19} + \Sigma_{102})/2 & S_{18} & = & (\Sigma_{71} + \Sigma_{28})/2 \\ S_7 & = & (\Sigma_{76} + \Sigma_{85})/2 & S_{19} & = & (\Sigma_{96} + \Sigma_{105})/2 \\ S_8 & = & (\Sigma_{74} + \Sigma_{38})/2 & S_{20} & = & (\Sigma_{94} + \Sigma_{310})/2 \\ S_9 & = & (\Sigma_{27} + \Sigma_{81})/2 & S_{21} & = & (\Sigma_{29} + \Sigma_{101})/2 \\ S_{10} & = & (\Sigma_{69} + \Sigma_{105})/2 & S_{22} & = & (\Sigma_{76} + \Sigma_{58})/2 \\ S_{11} & = & (\Sigma_{49} + \Sigma_{310})/2 & S_{23} & = & (\Sigma_{74} + \Sigma_{83})/2 \\ S_{12} & = & (\Sigma_{92} + \Sigma_{101})/2 & S_{24} & = & (\Sigma_{27} + \Sigma_{18})/2 \end{array} \tag{3.10}$$

Here $Tr(S_i S_j) = 4\,\delta_{ij}$. The gauge fields entering the covariant derivative D_μ are collected in the so called gauge term which was introduced in eq. (1.5). The above given conventions satisfy the following expansion

$$\begin{aligned} +i\frac{g}{\sqrt{2}} W^{ab}\,\Sigma_{ab} &= +i\frac{g}{\sqrt{2}}(V\cdot U + W_L\cdot L + W_R\cdot R + \mathbb{V}\cdot S) \\ &= +i\frac{g}{\sqrt{2}}\left[\begin{array}{cc} \Lambda_{11} & \Lambda_{12} \\ \Lambda_{21} & \Lambda_{22} \end{array}\right] \end{aligned} \tag{3.11}$$

where the real vector fields W_{ab} are the 45 gauge fields of $SO(10)$ with $a, b = (1, \ldots, 10)$. Here Λ_{ij} are 16×16 entries. The values of these entries in the above matrix vary with the representation that is used for Σ. Some examples will be given in § 4. The gauge fields are antisymmetric with respect to their group indices. i.e., $W_{ab} = -W_{ba}$ and they carry a 4-vector index μ, which is not explicitly shown. Using the above definitions, the gauge term can also be expressed in terms of the physical gauge fields. We have

$$\begin{aligned} + i\frac{g}{\sqrt{2}} W^{ab}\,\Sigma_{ab} = +i\,g\sqrt{2}\,\Bigg\{ & G\cdot U_G + (\,X_\alpha\cdot U_{X_\alpha} + h.c\,) + \sqrt{\frac{3}{2}}\frac{X_{B-L}}{\sqrt{2}}\cdot\frac{U_{B-L}}{2} + W_L^{\pm} L_{\pm} + W_R^{\pm} R_{\pm} \\ & + \frac{W_L^0}{\sqrt{2}} L_0 + \frac{W_R^0}{\sqrt{2}} R_0 + (D_{A_\alpha}\cdot A_\alpha + D_{A'_\alpha}\cdot A'_\alpha + D_{Y_\alpha}\cdot Y_\alpha + D_{Y'_\alpha}\cdot Y'_\alpha + h.c.)\Bigg\} \end{aligned} \tag{3.12}$$

The organization of the 45 real vector fields W_{ab} as described in the above gauge term into complex vector fields yield the following equations. The 8 gluons fields in terms of W_{ab}'s are expressed as

$$\begin{array}{ll} \bar{G}_4 = G_1 = & (W_{45} + W_{36} + i\,W_{53} + i\,W_{46})/2 \\ \bar{G}_5 = G_2 = & (W_{52} + W_{61} + i\,W_{62} + i\,W_{15})/2 \\ \bar{G}_6 = G_3 = & (W_{23} + W_{41} + i\,W_{31} + i\,W_{42})/2 \\ \bar{G}_7 = G_7 = & (W_{21} + W_{43} + 2\,W_{65})/\sqrt{6} \\ \bar{G}_8 = G_8 = & (W_{21} - 2\,W_{43} - W_{65})/\sqrt{6} \end{array} \tag{3.13}$$

Here G^1, G^2 and G^3 are conjugated to G^4, G^5 and G^6 respectively. G^7 and G^8 are made of diagonal elements. The lepto-quark gauge fields X_α, the X_{B-L} field and the $W_L^{\pm,0}$ and $W_R^{\pm,0}$ fields in terms of W_{ab}'s are given as

$$\begin{array}{llllll} {W_L}^{\pm} & = & (W_{98} + W_{107} \pm iW_{79} \pm iW_{108})/2 & X_r = X_1 & = & (W_{23} + W_{14} + iW_{31} + iW_{24})/2 \\ {W_L}^{0} & = & (W_{87} + W_{109})/\sqrt{2} & X_g = X_2 & = & (W_{25} + W_{61} + iW_{51} + iW_{62})/2 \\ {W_R}^{\pm} & = & (W_{98} + W_{710} \pm iW_{79} \pm iW_{810})/2 & X_b = X_3 & = & (W_{45} + W_{63} + iW_{53} + iW_{64})/2 \\ {W_R}^{0} & = & (W_{87} + W_{910})/\sqrt{2} & X_{B-L} & = & (W_{21} + W_{43} - W_{65})/\sqrt{3} \end{array} \tag{3.14}$$

The A_α, A'_α, Y_α and Y'_α gauge fields of the 45 in terms of the W_{ab}'s are given as

$$\begin{array}{llllll} A_r & = & (W_{75} + W_{68} + iW_{76} + iW_{85})/2 & Y_r & = & (W_{95} + W_{610} + iW_{96} + iW_{105})/2 \\ A_g & = & (W_{37} + W_{48} + iW_{74} + iW_{38})/2 & Y_g & = & (W_{39} + W_{410} + iW_{94} + iW_{310})/2 \\ A_b & = & (W_{71} + W_{82} + iW_{27} + iW_{81})/2 & Y_b & = & (W_{91} + W_{102} + iW_{29} + iW_{101})/2 \\ \\ A'_r & = & (W_{59} + W_{610} + iW_{69} + iW_{105})/2 & Y'_r & = & (W_{75} + W_{86} + iW_{76} + iW_{58})/2 \\ A'_g & = & (W_{93} + W_{410} + iW_{49} + iW_{310})/2 & Y'_g & = & (W_{37} + W_{84} + iW_{74} + iW_{83})/2 \\ A'_b & = & (W_{19} + W_{102} + iW_{92} + iW_{101})/2 & Y'_b & = & (W_{71} + W_{28} + iW_{27} + iW_{18})/2 \end{array} \tag{3.15}$$

3.2 The Charges of the 45 Fields

It can be shown that the raising and lowering generators defined in terms of Σ_{ab} in the former section satisfy a series of commutation relations by using again the Lie algebra of Σ_{ab}. These commutation relations reveal the charges of the related gauge fields. First we consider the A_α and the A'_α gauge fields together with the Y_α and the Y'_α gauge fields. We have

$$\begin{array}{lrcllrcl}
A_\alpha: & [D_i, L_3] & = & +\frac{1}{2}\, D_i \ , & & [D_i, R_3] & = & +\frac{1}{2}\, D_i \\
A'_\alpha: & [D_{i+3}, L_3] & = & -\frac{1}{2}\, D_{i+3} \ , & & [D_{i+3}, R_3] & = & +\frac{1}{2}\, D_{i+3} \\
Y_\alpha: & [D_{i+12}, L_3] & = & +\frac{1}{2}\, D_{i+12} \ , & & [D_{i+12}, R_3] & = & -\frac{1}{2}\, D_{i+12} \\
Y'_\alpha: & [D_{i+15}, L_3] & = & -\frac{1}{2}\, D_{i+15} \ , & & [D_{i+15}, R_3] & = & -\frac{1}{2}\, D_{i+15}
\end{array} \tag{3.16}$$

These equations hold for $i = (1, 2, 3)$. It is seen that these gauge fields carry simultaneously left and right isospin charges and decompose into a bi-doublet. Indeed we have arranged the S_i basis such that the well ordered output is obtained. The color charges of the A_α and the A'_α gauge fields follow as:

$$\begin{array}{lrcllrcl}
A_r: & [D_1, U_3] & = & +1/2\, D_1 & & [D_1, U_8] & = & +1/2\sqrt{3}\, D_1 \\
A_g: & [D_2, U_3] & = & -1/2\, D_2 & & [D_2, U_8] & = & +1/2\sqrt{3}\, D_2 \\
A_b: & [D_3, U_3] & = & 0\, D_3 & & [D_3, U_8] & = & -1/\sqrt{3}\, D_3 \\
\\
A'_r: & [D_4, U_3] & = & +1/2\, D_4 & & [D_4, U_8] & = & +1/2\sqrt{3}\, D_4 \\
A'_g: & [D_5, U_3] & = & -1/2\, D_5 & & [D_5, U_8] & = & +1/2\sqrt{3}\, D_5 \\
A'_b: & [D_6, U_3] & = & 0\, D_6 & & [D_6, U_8] & = & -1/\sqrt{3}\, D_6
\end{array} \tag{3.17}$$

From the above commutators, it is seen that the A_α and the A'_α gauge fields decompose into color triplets in the $U_3 - U_8$ space. The color charges of the Y_α and the Y'_α gauge fields follow as:

$$\begin{array}{lrcllrcl}
Y_r: & [D_{13}, U_3] & = & +1/2\, D_{13} & & [D_{13}, U_8] & = & +1/2\sqrt{3}\, D_{13} \\
Y_g: & [D_{14}, U_3] & = & -1/2\, D_{14} & & [D_{14}, U_8] & = & +1/2\sqrt{3}\, D_{14} \\
Y_b: & [D_{15}, U_3] & = & 0\, D_{15} & & [D_{15}, U_8] & = & -1/\sqrt{3}\, D_{15} \\
\\
Y'_r: & [D_{16}, U_3] & = & +1/2\, D_{16} & & [D_{16}, U_8] & = & +1/2\sqrt{3}\, D_{16} \\
Y'_g: & [D_{17}, U_3] & = & -1/2\, D_{17} & & [D_{17}, U_8] & = & +1/2\sqrt{3}\, D_{17} \\
Y'_b: & [D_{18}, U_3] & = & 0\, D_{18} & & [D_{18}, U_8] & = & -1/\sqrt{3}\, D_{18}
\end{array} \tag{3.18}$$

where Y_α and Y'_α also decompose into color triplets. The electric-charge eigenvalue operator is given through $Q = \sqrt{2/3}\, U_{15} + L_3 + R_3$ and applies to all known matter fields in $SO(10)$. The electric charges of the above gauge fields are found as

$$\begin{array}{lrcllrcl}
A_\alpha: & [D_i, Q] & = & +\frac{2}{3}\, D_i & & [D_i, (B-L)] & = & -\frac{2}{3}\, D_i \\
A'_\alpha: & [D_{i+3}, Q] & = & -\frac{1}{3}\, D_{i+3} & & [D_{i+3}, (B-L)] & = & -\frac{2}{3}\, D_{i+3} \\
Y_\alpha: & [D_{i+12}, Q] & = & -\frac{1}{3}\, D_{i+12} & & [D_{i+12}, (B-L)] & = & -\frac{2}{3}\, D_{i+12} \\
Y'_\alpha: & [D_{i+15}, Q] & = & -\frac{4}{3}\, D_{i+15} & & [D_{i+15}, (B-L)] & = & -\frac{2}{3}\, D_{i+15}
\end{array} \tag{3.19}$$

where $i = (1, 2, 3)$. Let us proceed with the gauge fields of $SU(3)$ by investigating the following commutators. We have

$$\begin{array}{rcllrcl}
[\,U_{G_1}, U_3] & = & +1\ \ U_{G_1} & & [\,U_{G_1}, U_8] & = & 0\ \ \ U_{G_1} \\
[\,U_{G_2}, U_3] & = & +1/2\, U_{G_2} & & [\,U_{G_2}, U_8] & = & +3/2\sqrt{3} U_{G_2} \\
[\,U_{G_3}, U_3] & = & -1/2\ \ U_{G_3} & & [\,U_{G_3}, U_8] & = & +3/2\sqrt{3} U_{G_3} \\
[\,U_{G_4}, U_3] & = & -1\ \ U_{G_4} & & [\,U_{G_4}, U_8] & = & 0\ \ \ U_{G_4} \\
[\,U_{G_5}, U_3] & = & -1/2\ \ U_{G_5} & & [\,U_{G_5}, U_8] & = & -3/2\sqrt{3} U_{G_5} \\
[\,U_{G_6}, U_3] & = & +1/2\ \ U_{G_6} & & [\,U_{G_6}, U_8] & = & -3/2\sqrt{3} U_{G_6} \\
[\,U_{G_7}, U_3] & = & 0\ \ \ U_{G_7} & & [\,U_{G_7}, U_8] & = & 0\ \ \ U_{G_7} \\
[\,U_{G_8}, U_3] & = & 0\ \ \ U_{G_8} & & [\,U_{G_8}, U_8] & = & 0\ \ \ U_{G_8}
\end{array} \tag{3.20}$$

It is seen that the gluons decompose into a color octet in the $U_3 - U_8$ space. It can be shown that all of the U_G generators commute with Q, L_3, R_3 and $(B - L)$. Consequently, gluons carry only *color* charge. The U_X generators commute with L_3 and R_3 and carry neither left nor right isospin. But they carry color. We have

$$\begin{array}{lrcllrcl}
X_r: & [\,U_{X_1}, U_3] & = & +1/2\, U_{X_1} & & [\,U_{X_1}, U_8] & = & +1/2\sqrt{3}\, U_{X_1} \\
X_g: & [\,U_{X_2}, U_3] & = & -1/2\, U_{X_2} & & [\,U_{X_2}, U_8] & = & +1/2\sqrt{3}\, U_{X_2} \\
X_b: & [\,U_{X_3}, U_3] & = & 0\ \ \ U_{X_3} & & [\,U_{X_3}, U_8] & = & -1/\sqrt{3}\ \ U_{X_3}
\end{array} \tag{3.21}$$

The X_α gauge fields decompose into a color triplet isospin singlets. The electric and $B-L$ charges of the X_α are found as

$$\begin{array}{llcllcl} X_r: & [U_{X_1},Q] & = & \frac{2}{3}U_{X_1} & X_r: & [U_{X_1},(B-L)] & = & \frac{4}{3}U_{X_1} \\ X_g: & [U_{X_2},Q] & = & \frac{2}{3}U_{X_2} & X_g: & [U_{X_2},(B-L)] & = & \frac{4}{3}U_{X_2} \\ X_b: & [U_{X_3},Q] & = & \frac{2}{3}U_{X_3} & X_b: & [U_{X_3},(B-L)] & = & \frac{4}{3}U_{X_3} \end{array} \tag{3.22}$$

where Q and $B-L$ are defined as before. The gluons, the X_α's and the X_{B-L} fields form the $(15,1,1)$ multiplet with respect to the maximal subgroup of $SU(4)\times SU(2)_L\times SU(2)_R$. Finally we deal with the $W_L^{\pm,3}$ and $W_R^{\pm,3}$ gauge fields. They commute with all generators of $SU(4)$, thereby with $SU(3)$ and $B-L$ as well and carry no color and B-L charge. Their electric charges follow through

$$\begin{array}{llcllcl} W_L^+: & [L_+,Q] & = & +1\,L_+ & W_R^+: & [R_+,Q] & = & +1\,R_+ \\ W_L^3: & [L_3,Q] & = & 0\ L_3 & W_R^3: & [R_3,Q] & = & 0\ R_3 \\ W_L^-: & [L_-,Q] & = & -1\,L_- & W_R^-: & [R_-,Q] & = & -1\,R_- \end{array} \tag{3.23}$$

From the other side the $W_L^{\pm,0}$ and the $W_R^{\pm,0}$ gauge fields decompose into isospin triplet vectors. We have

$$\begin{array}{llcllcl} W_L^+: & [L^+,L_3] & = & +1\,L^+ & W_R^+: & [R^+,R_3] & = & +1\,R^+ \\ W_L^3: & [L_3,L_3] & = & 0\ L_3 & W_R^3: & [R_3,R_3] & = & 0\ R_3 \\ W_L^-: & [L^-,L_3] & = & -1\,L^- & W_R^-: & [R^-,R_3] & = & -1\,R^- \end{array} \tag{3.24}$$

They are singlets with respect to each other. All the above derived charges of the gauge fields in $SO(10)$ are summarized in Table 3.1.

Charges of the 45 Gauge Bosons

	Q	B-L	I_{3R}	I_{3L}	Y		Q	B-L	I_{3R}	I_{3L}	Y
A_r	2/3	-2/3	+1/2	+1/2	1/3	G_i	0	0	0	0	0
A_g	2/3	-2/3	+1/2	+1/2	1/3	X_{B-L}	0	0	0	0	0
A_b	2/3	-2/3	+1/2	+1/2	1/3	X_r	2/3	4/3	0	0	2/3
A'_r	-1/3	-2/3	+1/2	-1/2	1/3	X_g	2/3	4/3	0	0	2/3
A'_g	-1/3	-2/3	+1/2	-1/2	1/3	X_b	2/3	4/3	0	0	2/3
A'_b	-1/3	-2/3	+1/2	-1/2	1/3	W_L^+	+1	0	0	+1	0
Y_r	-1/3	-2/3	-1/2	+1/2	-5/3	W_L^0	0	0	0	0	0
Y_g	-1/3	-2/3	-1/2	+1/2	-5/3	W_L^-	-1	0	0	-1	0
Y_b	-1/3	-2/3	-1/2	+1/2	-5/3	W_R^+	+1	0	+1	0	+1
Y'_r	-4/3	-2/3	-1/2	-1/2	-5/3	W_R^0	0	0	0	0	0
Y'_g	-4/3	-2/3	-1/2	-1/2	-5/3	W_R^-	-1	0	-1	0	-1
Y'_b	-4/3	-2/3	-1/2	-1/2	-5/3						

Tab. 3.1: Charges of the 45 Gauge Bosons

3.3 Weight Diagrams

The decompositions of the 45 gauge fields with respect to $SU_3 \times SU_{2L} \times SU_{2R} \times U_{1B-L}$ can be found using the former commutation relations in § (3.2) and can also be directly read off from Table (3.1). We see that A_α, A'_α, Y_α and Y'_α are all color triplets forming a bi-doublet. Their charge conjugates are color anti-triplets being a bi-doublet as well. The triplet and the anti-triplet is shown in the lower part of Fig. (3.1). These pairs are distinguished by their $B-L$ charges. From the other side, the X_α fields form a triplet and the charge conjugated $\bar{X}_\alpha$ fields form an anti-triplet. These are also distinguished by their $B-L$ charges and are shown in the upper part of Fig. (3.1). In the same figure X_{B-L} and the gluons are corresponding to the singlet and the octet fields.

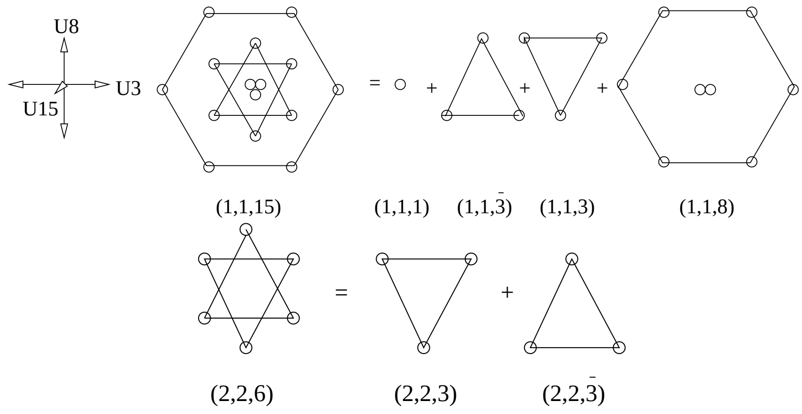

Fig. 3.1: The decomposition of 45 with respect to $SU_3 \times SU_{2L} \times SU_{2R} \times U_{1B-L}$. The L, R isospin weights are suppressed. U_{15} points out of page.

3.4 Decompositions of the 45

The 45 gauge fields decompose under $SU_4 \times SU_{2L} \times SU_{2R}$, $SU_3 \times SU_{2L} \times SU_{2R} \times U_{1B-L}$ and $SU_3 \times SU_{2L} \times U_{1Y}$ respectively as

$$
\begin{array}{ccccccccccccc}
45 & = & (15,1,1) & + & (6,2,2) & + & (1,3,1) & + & (1,1,3) & & \\[2ex]
45 & = & (8,1,1)_0 & + & (1,3,1)_0 & + & (1,1,3)_0 & + & (3,2,2)_{-2/3} & + & (\bar{3},2,2)_{2/3} \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
 & & G_i & & W_L^{\pm,0} & & W_R^{\pm,0} & & A_\alpha Y_\alpha & & \bar{Y}'_\alpha \bar{A}'_\alpha \\
 & & & & & & & & A'_\alpha Y'_\alpha & & \bar{Y}_\alpha \bar{A}_\alpha \\[1ex]
 & & (1,1,1)_0 & + & (3,1,1)_{2/3} & + & (\bar{3},1,1)_{-2/3} & & & & \\
 & & \downarrow & & \downarrow & & \downarrow & & & & \\
 & & X_{B-L} & & X_\alpha & & \bar{X}_\alpha & & & & \\[2ex]
45 & = & (8,1)_0 & + & (1,3)_0 & + & (1,1)_0 & + & (3,2)_{-5/3} & + & (\bar{3},2)_{5/3} \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
 & & G_i & & W_L^{\pm,0} & & X_{B-L} & & Y_\alpha, Y'_\alpha & & \bar{Y}'_\alpha, \bar{Y}_\alpha \\[1ex]
 & & +\,(3,2)_{1/3} & + & (\bar{3},2)_{-1/3} & + & (3,1)_{2/3} & + & (\bar{3},1)_{-2/3} & + & (1,1)_1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
 & & A_\alpha, A'_\alpha & & \bar{A}'_\alpha, \bar{A}_\alpha & & X_\alpha & & \bar{X}_\alpha & & W_R^+ \\[1ex]
 & & +\,(1,1)_0 & + & (1,1)_{-1} & & & & & & \\
 & & \downarrow & & \downarrow & & & & & & \\
 & & W_R^0 & & W_R^- & & & & & &
\end{array}
\tag{3.25}
$$

3.5 The $SU(5) \times U(1)$ content of $SO(10)$

3.5.1 Fields

In this section, we will not go into a formulation of the well known $SU(5)$ theory. But we will project out the $SU(5)$ theory from the $SO(10)$ theory for completeness and practical interest. At the end of this section, we will show which generators in $SO(10)$ make the $U(5)$ maximal subgroup. This procedure can be particularly useful when one considers to break the initial $SO(10)$ symmetry spontaneously down to the intermediate $SU(5)$ symmetry.

The 24 gauge field analysis in terms of the S_i basis given in eq. (3.10) is particularly fruitful because it contains some of the $SU(5)$ fields that are also $SO(10)$ fields. In any of our bases (Basis A, B or C), the $S_{13}, \ldots, S_{24}$ basis corresponds to 12 generators of the 24 dimensional $SU(5)$ group. These span also the gauge fields which lie exterior with respect to the maximal subgroup of $SU(5)$ which is the $SU_3 \times SU_{2L} \times U_{1Y}$ direct product gauge group. This maximal subgroup contains the 8 gluons, the $W_L^{\pm,0}$ gauge fields and finally the hypercharge gauge field X_Y. The eigenvalue operator for hypercharge is given through the linear combination $Y = R_3 + U_{B-L}/2$.

From Table (3.1), it is seen that the Y_α and Y'_α gauge fields of the $SU(5)$ theory when embedded in a richer $SO(10)$ theory appear to decompose into right isospin doublets as well as left. It is obvious at this point that the $SU(5)$ theory doesn't provide a full left-right symmetry. In this respect the A_α and the A'_α gauge fields are complementary to the Y_α and the Y'_α gauge fields and these appear as doublets under the decomposition of the 45 with respect to $SU(3) \times SU(2) \times U(1)_Y$ with a different hypercharge, respectively $1/3$ and $-1/3$ as shown in eq. (3.25). Let use analyze the 24 fields in $SU(5)$ whose known decomposition follows as

$$\begin{array}{ccccccccccc} 24 & = & (8,1)_0 & + & (1,3)_0 & + & (1,1)_0 & + & (3,2)_{-5/3} & + & (\bar{3},2)_{5/3} \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ & & G_i & & W_L^{\pm,0} & & X_Y & & Y_\alpha, Y'_\alpha & & \bar{Y}'_\alpha, \bar{Y}_\alpha \\ & & & & & & \text{mixing} & & & & \end{array} \tag{3.26}$$

Here the X_Y is a mixture of the two $(1,1)_0$ singlets which are the W_R^0 and X_{B-L} given in eq.(3.25). The $SU(5)$ generators will be useful. They are given in Table (3.2) [47]. The gauge term of the $SU(5)$ theory reads

$$\mathbf{U}_i = \left(\begin{array}{cccc|c} & & & & 0 \\ & SU_4 & & & 0 \\ & & & & 0 \\ & & & & 0 \\ \hline 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

$$\mathbf{U}_{16} = \frac{1}{2}\begin{pmatrix} 0&0&0&0&1\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\1&0&0&0&0 \end{pmatrix} \quad \mathbf{U}_{17} = \frac{1}{2}\begin{pmatrix} 0&0&0&0&-i\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\i&0&0&0&0 \end{pmatrix} \quad \mathbf{U}_{18} = \frac{1}{2}\begin{pmatrix} 0&0&0&0&0\\0&0&0&0&1\\0&0&0&0&0\\0&0&0&0&0\\0&1&0&0&0 \end{pmatrix}$$

$$\mathbf{U}_{19} = \frac{1}{2}\begin{pmatrix} 0&0&0&0&0\\0&0&0&0&-i\\0&0&0&0&0\\0&0&0&0&0\\0&i&0&0&0 \end{pmatrix} \quad \mathbf{U}_{20} = \frac{1}{2}\begin{pmatrix} 0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&1\\0&0&0&0&0\\0&0&1&0&0 \end{pmatrix} \quad \mathbf{U}_{21} = \frac{1}{2}\begin{pmatrix} 0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&-i\\0&0&0&0&0\\0&0&i&0&0 \end{pmatrix}$$

$$\mathbf{U}_{22} = \frac{1}{2}\begin{pmatrix} 0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&1\\0&0&0&1&0 \end{pmatrix} \quad \mathbf{U}_{23} = \frac{1}{2}\begin{pmatrix} 0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&-i\\0&0&0&i&0 \end{pmatrix} \quad \mathbf{U}_{24} = \frac{1}{2}\frac{1}{\sqrt{10}}\begin{pmatrix} 1&0&0&0&0\\0&1&0&0&0\\0&0&1&0&0\\0&0&0&1&0\\0&0&0&0&-4 \end{pmatrix}$$

Tab. 3.2: The $SU(5)$ generators in the fundamental form. The representation is normalized to $Tr(\mathbf{U}_i\mathbf{U}_j) = \frac{1}{2}\delta_{ij}$ and the generators obey $[\mathbf{U}_k, \mathbf{U}_\mu] = i f_{k\mu\nu}\mathbf{U}_\nu$, there is summation over ν, where $k, \mu, \nu = 1, \ldots, 24$ and $i = 1, \ldots, 15$. The $SU(4)$ part of these structure constants are tabulated in Table (2.1)

$$\mathbf{U}\cdot\mathbf{V} = \frac{1}{\sqrt{2}}\begin{pmatrix} \lambda_1 & G_4 & G_5 & \bar{Y}'_r & \bar{Y}_r \\ G_1 & \lambda_2 & G_6 & \bar{Y}'_g & \bar{Y}_g \\ G_2 & G_3 & \lambda_3 & \bar{Y}'_b & \bar{Y}_b \\ Y'_r & Y'_g & Y'_b & \lambda_4 & W_L^- \\ Y_r & Y_g & Y_b & W_L^+ & \lambda_5 \end{pmatrix} \tag{3.27}$$

where we have defined 25 real vector fields $\mathbf{V}^i_\mu$ to get the adjoint representation of the gauge boson matrix. The $SU(3)$ gauge fields are

$$\begin{aligned} G_1 &= \bar{G}_4 = (\mathbf{V}_1 + i\mathbf{V}_2)/\sqrt{2} & \mathbf{U}_{G_1} &= \mathbf{U}^\dagger_{G_4} = (\mathbf{U}_1 - i\mathbf{U}_2)/2 \\ G_2 &= \bar{G}_5 = (\mathbf{V}_4 + i\mathbf{V}_5)/\sqrt{2} & \mathbf{U}_{G_2} &= \mathbf{U}^\dagger_{G_5} = (\mathbf{U}_4 - i\mathbf{U}_5)/2 \\ G_3 &= \bar{G}_6 = (\mathbf{V}_6 + i\mathbf{V}_7)/\sqrt{2} & \mathbf{U}_{G_3} &= \mathbf{U}^\dagger_{G_6} = (\mathbf{U}_6 - i\mathbf{U}_7)/2 \\ G_7 &= \bar{G}_7 = (\mathbf{V}_3\sqrt{3} + \mathbf{V}_8)/2 & \mathbf{U}_{G_7} &= (\mathbf{U}_3/\sqrt{3} + \mathbf{U}_8)/\sqrt{2} \\ G_8 &= \bar{G}_8 = (-\mathbf{V}_3\sqrt{3} + \mathbf{V}_8)/2 & \mathbf{U}_{G_8} &= (-\mathbf{U}_3/\sqrt{3} + \mathbf{U}_8)/\sqrt{2} \end{aligned} \tag{3.28}$$

The diagonal entries in the gauge term matrix above are

$$\begin{aligned} \lambda_1 &= \frac{2G_7}{\sqrt{6}} - \frac{2X_Y}{\sqrt{30}} & \lambda_4 &= \frac{W_L^3}{\sqrt{2}} + \frac{3X_Y}{\sqrt{30}} \\ \lambda_2 &= \frac{2G_8}{\sqrt{6}} - \frac{2X_Y}{\sqrt{30}} & \lambda_5 &= -\frac{W_L^3}{\sqrt{2}} + \frac{3X_Y}{\sqrt{30}} \\ \lambda_3 &= -\frac{2G_7}{\sqrt{6}} - \frac{2G_8}{\sqrt{6}} - \frac{2X_Y}{\sqrt{30}} \end{aligned} \tag{3.29}$$

The $SU(2)_L$ generators are

$$\mathbf{L}_1 = \mathbf{U}_{22} \quad , \quad \mathbf{L}_2 = \mathbf{U}_{23} \quad , \quad \mathbf{L}_3 = \frac{\sqrt{10}}{4}\mathbf{U}_{24} - \frac{\sqrt{6}}{4}\mathbf{U}_{15} = \mathrm{diag}(0,0,0,+\frac{1}{2},-\frac{1}{2}) \tag{3.30}$$

and the $SU(2)_L$ gauge fields are

$$W_L^\pm = \sqrt{\frac{1}{2}}(\mathbf{V}_{22} \pm \ i\ \mathbf{V}_{23}) \quad , \quad W_L^3 = -\sqrt{\frac{3}{8}}\mathbf{V}_{15} + \sqrt{\frac{5}{8}}\mathbf{V}_{24} \tag{3.31}$$

The $U(1)_Y$ hypercharge generator and the X_Y hypercharge gauge field is

$$\mathbf{Y} = -\sqrt{\frac{3}{5}}\left(\frac{\sqrt{10}}{4}\mathbf{U}_{24} + \frac{5\sqrt{6}}{12}\mathbf{U}_{15}\right) = \sqrt{\frac{3}{5}}\mathrm{diag}(-\frac{1}{3},-\frac{1}{3},-\frac{1}{3},\frac{1}{2},\frac{1}{2})\,, \quad X_Y = -\sqrt{\frac{5}{8}}\mathbf{V}_{15} - \sqrt{\frac{3}{8}}\mathbf{V}_{24} \tag{3.32}$$

The charge generator Q is by definition made to fit the 5 as below. If it is applied to the conjugate i.e, $\bar{5}$, Q should reverse sign.

$$Q = \mathbf{L}_3 + \sqrt{\frac{5}{3}}\mathbf{Y} = -2\sqrt{\frac{2}{3}}\mathbf{U}_{15} = diag(-\frac{1}{3},-\frac{1}{3},-\frac{1}{3},1,0) \tag{3.33}$$

The Y_α and Y'_α gauge bosons are

$$\begin{aligned} Y_r &= (\mathbf{V}_9 + i\mathbf{V}_{10})/\sqrt{2} & D_{16} &= D^\dagger_{22} \equiv \mathbf{U}_{Y_r} = (\mathbf{U}_9 - i\mathbf{U}_{10})/2 \\ Y_g &= (\mathbf{V}_{11} + i\mathbf{V}_{12})/\sqrt{2} & D_{17} &= D^\dagger_{23} \equiv \mathbf{U}_{Y_g} = (\mathbf{U}_{11} - i\mathbf{U}_{12})/2 \\ Y_b &= (\mathbf{V}_{13} + i\mathbf{V}_{14})/\sqrt{2} & D_{18} &= D^\dagger_{24} \equiv \mathbf{U}_{Y_b} = (\mathbf{U}_{13} - i\mathbf{U}_{14})/2 \\ \\ Y'_r &= (\mathbf{V}_{16} + i\mathbf{V}_{17})/\sqrt{2} & D_{13} &= D^\dagger_{19} \equiv \mathbf{U}_{Y'_r} = (\mathbf{U}_{16} - i\mathbf{U}_{17})/2 \\ Y'_g &= (\mathbf{V}_{18} + i\mathbf{V}_{19})/\sqrt{2} & D_{14} &= D^\dagger_{20} \equiv \mathbf{U}_{Y'_g} = (\mathbf{U}_{18} - i\mathbf{U}_{19})/2 \\ Y'_b &= (\mathbf{V}_{20} + i\mathbf{V}_{21})/\sqrt{2} & D_{15} &= D^\dagger_{21} \equiv \mathbf{U}_{Y'_b} = (\mathbf{U}_{20} - i\mathbf{U}_{21})/2 \end{aligned} \tag{3.34}$$

Here $D_{13},\ldots,D_{24}$ are the raising/lowering generators of the $SO(10)$ theory which correctly match the $\mathbf{U}_{Y_\alpha}$ and $\mathbf{U}_{Y'_\alpha}$ of the $SU(5)$ subgroup. Note that the Y_α and Y'_α gauge bosons above form a color triplet and posses the same charges as given in Table (3.1). From the other side the $\mathbf{U}^\dagger_{G_i}$ and $\mathbf{V}_i$ from above match those in eq. (3.1). This matching also holds for the isospin and hypercharge sectors.

3.5.2 Generators

Owing to the correspondence among the fields achieved on basis of their charges and decompositions in the previous section, it becomes possible to sort out the following set of generators on the left hand side which exist in $SO(10)$. These generators can be matched with the $\mathbf{U}_i$ of $SU(5)$ for $i=(1,\ldots,24)$ given in Table 3.2 as shown on the right hand side below. Note that all generators are normalized to $Tr(\mathbf{U}_i\mathbf{U}_j)=4\delta_{ij}$. The correspondence established between the two sets exhibits the property that they both satisfy the same Lie algebra given through $[\mathbf{U}_k,\mathbf{U}_\mu]=if_{k\mu\nu}\mathbf{U}_\nu$.

$$
\begin{aligned}
U_1 &= (\Sigma_{45}+\Sigma_{36})/2 \\
U_2 &= (\Sigma_{53}+\Sigma_{46})/2 \\
U_3 &= (\Sigma_{65}+\Sigma_{43})/2 \\
U_4 &= (\Sigma_{52}+\Sigma_{61})/2 \\
U_5 &= (\Sigma_{51}+\Sigma_{62})/2 \\
U_6 &= (\Sigma_{23}+\Sigma_{41})/2 \\
U_7 &= (\Sigma_{31}+\Sigma_{24})/2 \\
U_8 &= (2\Sigma_{21}+\Sigma_{34}+\Sigma_{65})/(2\sqrt{3}) \\
\\
S_{13} &= (\Sigma_{95}+\Sigma_{610})/2 \\
S_{14} &= (\Sigma_{39}+\Sigma_{410})/2 \\
S_{15} &= (\Sigma_{91}+\Sigma_{102})/2 \\
S_{16} &= (\Sigma_{75}+\Sigma_{86})/2 \\
S_{17} &= (\Sigma_{37}+\Sigma_{84})/2 \\
S_{18} &= (\Sigma_{71}+\Sigma_{28})/2 \\
S_{19} &= (\Sigma_{96}+\Sigma_{105})/2 \\
S_{20} &= (\Sigma_{94}+\Sigma_{310})/2 \\
S_{21} &= (\Sigma_{29}+\Sigma_{101})/2 \\
S_{22} &= (\Sigma_{76}+\Sigma_{58})/2 \\
S_{23} &= (\Sigma_{74}+\Sigma_{83})/2 \\
S_{24} &= (\Sigma_{27}+\Sigma_{18})/2 \\
\\
L_1 &= (\Sigma_{79}+\Sigma_{108})/2 \\
L_2 &= (\Sigma_{98}+\Sigma_{107})/2 \\
L_3 &= (\Sigma_{87}+\Sigma_{109})/2 \\
\\
Y' &= \tfrac{1}{2}\sqrt{\tfrac{3}{5}}(\Sigma_{87}+\Sigma_{910}+\tfrac{2}{3}\Sigma_{21}+\tfrac{2}{3}\Sigma_{43}-\tfrac{2}{3}\Sigma_{65})
\end{aligned}
\quad\Rightarrow\quad
\begin{aligned}
\mathbf{U}_1 &\equiv U_1 \\
\mathbf{U}_2 &\equiv U_2 \\
\mathbf{U}_3 &\equiv U_3 \\
\mathbf{U}_4 &\equiv U_4 \\
\mathbf{U}_5 &\equiv U_5 \\
\mathbf{U}_6 &\equiv U_6 \\
\mathbf{U}_7 &\equiv U_7 \\
\mathbf{U}_8 &\equiv U_8 \\
\mathbf{U}_9 &\equiv S_{16} \\
\mathbf{U}_{10} &\equiv S_{22} \\
\mathbf{U}_{11} &\equiv S_{17} \\
\mathbf{U}_{12} &\equiv S_{23} \\
\mathbf{U}_{13} &\equiv S_{18} \\
\mathbf{U}_{14} &\equiv S_{24} \\
\mathbf{U}_{15} &\equiv -\sqrt{\tfrac{3}{2}}(\tfrac{L_3}{2}+\sqrt{\tfrac{5}{3}}\tfrac{Y'}{2}) \\
\mathbf{U}_{16} &\equiv S_{13} \\
\mathbf{U}_{17} &\equiv S_{19} \\
\mathbf{U}_{18} &\equiv S_{14} \\
\mathbf{U}_{19} &\equiv S_{20} \\
\mathbf{U}_{20} &\equiv S_{15} \\
\mathbf{U}_{21} &\equiv S_{21} \\
\mathbf{U}_{22} &\equiv L_1 \\
\mathbf{U}_{23} &\equiv L_2 \\
\mathbf{U}_{24} &\equiv \tfrac{5}{\sqrt{10}}(\tfrac{L_3}{2}-\sqrt{\tfrac{3}{5}}\tfrac{Y'}{2})
\end{aligned}
\tag{3.35}
$$

The hypercharge generator Y' follows from the hypercharge eigenvalue operator $R_3+(B-L)/2$ and is above normalized to $Tr[Y'.Y']=4$. A technical detail that one encounters here is to properly label the direct product of $SU(2)\times SU(2)$, since the one that exists in $SU(5)$, should be labelled with L and consequently the other with R. The maximal subgroup of $SO(10)$ was identified as $U(5)=SU(5)\times U(1)$, therefore the 25^{th} generator of the $U(1)$ part should commute with all the above generators and is easy to identify from the charges summarized in Table (3.1). We have

$$\mathbf{U}_{25} = \frac{4}{\sqrt{10}}(\frac{R_3}{2}-\frac{3}{4}\frac{B-L}{2}) \tag{3.36}$$

where $Tr[\,U_{25}\cdot U_{25}]=4$. We close this section with the following remark: It is also possible to study the $SU(5)$ theory by using the above selected $SO(10)$ representation. In this case the generator $\mathbf{U}_{25}$ will be related with the global conservation of $B-L$ [64]. Furthermore the R_3 component of $\mathbf{U}_{25}$ should be omitted. Because in $SU(5)$ there exists no right-isospin. Also note that in the fundamental representation of $SU(5)$ there is no way to define a traceless $B-L$ generator whereas in $SO(10)$ this is possible. It is remarkable to see how the global conservation of $B-L$ in $SU(5)$ can be recovered when it is studied through the $SO(10)$ representation.

4. THE GAUGE TERM: (45)

In contrast to the electroweak theory, it is a rather sophisticated and an exhausting task to illustrate the content of the $SO(10)$ gauge term. In this section, we make use of the definitions and conventions of the general scheme that we adapted in §3 in order to depict the gauge term by means of the following expansion

$$
\begin{aligned}
+i\frac{g}{\sqrt{2}}W^{ab}\,\Sigma_{ab} = +i\,g\,\sqrt{2}\,\Bigg\{ & G\cdot U_G + (\,X_\alpha\cdot U_{X_\alpha} + h.c.) + \frac{\sqrt{3}}{4}X_{B-L}\cdot U_{B-L} + W_L^{\pm}L_{\pm} + \frac{W_L^0}{\sqrt{2}}L_0 \\
& + W_R^{\pm}R_{\pm} + \frac{W_R^0}{\sqrt{2}}R_0 + (D_{A_\alpha}\cdot A_\alpha + D_{A'_\alpha}\cdot A'_\alpha + D_{Y_\alpha}\cdot Y_\alpha + D_{Y'_\alpha}\cdot Y'_\alpha + h.c.)\Bigg\} \\
= +i\frac{g}{\sqrt{2}}\left[\begin{array}{cc} \Lambda_{11} & \Lambda_{12} \\ \Lambda_{21} & \Lambda_{22} \end{array}\right] &
\end{aligned}
\tag{4.1}
$$

The expansion holds universally. However the entries $\Lambda_{11}, \Lambda_{12}, \Lambda_{21}$ and Λ_{22} being matrices each of size 16×16 explicitly depend on the basis. In the next 3 sections, we investigate their explicit form by applying the previously introduced Γ bases in § 2.3 to the above expansion.

4.1 Basis A

In this basis, Λ_{12} and Λ_{21} are occupied by zeros. The Λ_{11} and Λ_{22} parts of the gauge term in eq. (4.1) are respectively given as below. The empty spots in these matrices also correspond to single zero entries and are left blank for a clear appearance. The resulting fermion assignment of the Ψ_L and Ψ_R determined by these set of generators are

$$
\Psi_L = \left[\begin{array}{c} u_r \\ \nu \\ u_g \\ u_b \\ -\nu^c \\ -u_r^c \\ -u_g^c \\ -u_b^c \\ d_r \\ e \\ d_g \\ d_b \\ e^c \\ d_r^c \\ d_g^c \\ d_b^c \end{array}\right]_L \quad , \quad \Psi_R = \left[\begin{array}{c} u_r \\ \nu \\ u_g \\ u_b \\ -\nu^c \\ -u_r^c \\ -u_g^c \\ -u_b^c \\ d_r \\ e \\ d_g \\ d_b \\ e^c \\ d_r^c \\ d_g^c \\ d_b^c \end{array}\right]_R \quad , \quad \Psi = \left(\begin{array}{c} \Psi_L \\ \Psi_R \end{array}\right)
\tag{4.2}
$$

and can be obtained through the electric-charge eigenvalue operator $Q = (1/2)\,U_{B-L} + L_3 + R_3$ whose components are given in eq. (3.3) and eq. (3.9). The spinors Ψ_L and Ψ_R couple to the below matrices Λ_{11} and Λ_{22} respectively. This is indicated by the indices L and R which are attached to these matrices above. The total spinor Ψ is given in the last term in the above expression. It should be noted that every single fermion in Ψ_L is understood to be a left handed spinor and every single fermion in Ψ_R is understood to be a right handed spinor.

$$
\begin{bmatrix}
\lambda_1 & -\bar{X}_r & -G_4 & G_5 & \bar{A}_r & 0 & Y'_g & -Y'_b & W_L^- & & & & \bar{Y}_r & 0 & -A'_g & A'_b \\
-X_r & \lambda_2 & X_g & -X_b & 0 & \bar{A}_r & \bar{A}_b & \bar{A}_g & & W_L^- & & & 0 & -\bar{Y}_r & -\bar{Y}_b & -\bar{Y}_g \\
-G_1 & \bar{X}_g & \lambda_3 & G_6 & \bar{A}_g & -Y'_b & Y'_r & 0 & & & W_L^- & & -\bar{Y}_g & A'_b & -A'_r & 0 \\
G_2 & -\bar{X}_b & -G_3 & \lambda_4 & \bar{A}_b & -Y'_g & 0 & Y'_r & & & & W_L^- & \bar{Y}_b & A'_g & 0 & -A'_r \\
A_r & 0 & -A_g & A_b & \lambda_5 & \bar{X}_r & \bar{X}_b & -\bar{X}_g & A'_r & 0 & -A'_g & A'_b & W_R^+ & & & \\
0 & A_r & \bar{Y}'_b & \bar{Y}'_g & -X_r & \lambda_6 & -G_2 & -G_1 & 0 & A'_r & \bar{Y}_g & \bar{Y}_b & & W_R^+ & & \\
\bar{Y}'_g & A_b & \bar{Y}'_r & 0 & -X_b & -G_5 & \lambda_7 & -G_6 & -\bar{Y}_b & A'_b & -\bar{Y}_r & 0 & & & W_R^+ & \\
\bar{Y}'_b & A_g & 0 & \bar{Y}'_r & -X_g & -G_4 & -G_3 & \lambda_8 & \bar{Y}_g & A'_g & 0 & -\bar{Y}_r & & & & W_R^+ \\
W_L^+ & & & & -\bar{A}'_r & 0 & Y_b & Y_g & \lambda_9 & -\bar{X}_r & -G_4 & G_5 & -\bar{Y}'_r & 0 & A_g & -A_g \\
 & W_L^+ & & & 0 & \bar{A}'_r & \bar{A}'_g & \bar{A}'_b & -X_r & \lambda_{10} & X_g & -X_b & 0 & -\bar{Y}'_r & -\bar{Y}'_b & -\bar{Y}'_g \\
 & & W_L^+ & & \bar{A}'_g & Y_b & -Y_r & 0 & -G_1 & \bar{X}_g & \lambda_{11} & -G_6 & \bar{Y}'_g & -A_b & A_r & 0 \\
 & & & W_L^+ & -\bar{A}'_b & Y_g & 0 & -Y_r & G_2 & -\bar{X}_b & -G_3 & \lambda_{12} & -\bar{Y}'_b & -A_g & 0 & A_r \\
Y_r & 0 & -Y_b & Y_g & W_R^- & & & & Y'_r & 0 & -Y'_g & Y'_b & \lambda_{13} & -\bar{X}_r & -\bar{X}_b & -\bar{X}_g \\
0 & Y_r & -\bar{A}'_b & -\bar{A}'_g & & W_R^- & & & 0 & Y'_r & \bar{A}_b & \bar{A}_g & -X_r & \lambda_{14} & -G_2 & -G_1 \\
\bar{A}'_g & Y_b & \bar{A}'_r & 0 & & & W_R^- & & \bar{A}_g & Y'_b & -\bar{A}_r & 0 & -X_b & -G_1 & \lambda_{15} & -G_6 \\
-\bar{A}'_b & Y_g & 0 & \bar{A}'_r & & & & W_R^- & \bar{A}_b & Y'_g & 0 & -\bar{A}_r & -X_g & -G_4 & -G_3 & \lambda_{16}
\end{bmatrix}_L
$$

$$
\begin{bmatrix}
\lambda_{17} & -\bar{X}_r & -G_4 & G_5 & \bar{A}_r & 0 & Y'_g & -Y'_b & W_R^- & & & & \bar{A}'_r & 0 & Y_b & -Y_g \\
-X_r & \lambda_{18} & X_g & -X_b & 0 & \bar{A}_r & \bar{A}_b & \bar{A}_g & & W_R^- & & & 0 & \bar{A}'_r & \bar{A}'_g & \bar{A}'_b \\
-G_1 & \bar{X}_g & \lambda_{19} & G_6 & \bar{A}_g & -Y'_b & Y'_r & 0 & & & W_R^- & & -\bar{A}'_b & -Y_g & 0 & Y_r \\
G_2 & -\bar{X}_b & -G_3 & \lambda_{20} & \bar{A}_b & -Y'_g & 0 & Y'_r & & & & W_R^- & \bar{Y}_b & -A'_g & 0 & A'_r \\
A_r & 0 & -A_g & A_b & \lambda_{21} & \bar{X}_r & \bar{X}_b & -\bar{X}_g & -Y_r & 0 & Y_b & -Y_g & W_L^+ & & & \\
0 & A_r & \bar{Y}'_b & \bar{Y}'_g & -X_r & \lambda_{22} & -G_2 & -G_1 & 0 & -Y_r & \bar{A}'_b & \bar{A}'_g & & W_L^+ & & \\
\bar{Y}'_g & A_b & \bar{Y}'_r & 0 & -X_b & -G_5 & \lambda_{23} & -G_6 & -\bar{A}'_g & -Y_b & -\bar{A}'_r & 0 & & & W_L^+ & \\
\bar{Y}'_b & A_g & 0 & \bar{Y}'_r & -X_g & -G_4 & -G_3 & -\lambda_{24} & \bar{A}'_b & -Y_g & 0 & -\bar{A}'_r & & & & W_L^+ \\
W_R^+ & & & & \bar{Y}_r & 0 & A'_g & -A'_b & \lambda_{25} & -\bar{X}_r & -G_4 & G_5 & -\bar{Y}'_r & 0 & A_g & -A_g \\
 & W_R^+ & & & 0 & \bar{Y}_r & \bar{Y}_b & \bar{Y}_g & -X_r & \lambda_{26} & X_g & -X_b & 0 & -\bar{Y}'_r & -\bar{Y}'_b & -\bar{Y}'_g \\
 & & W_R^+ & & -\bar{Y}_g & -A'_b & A'_r & 0 & -G_1 & \bar{X}_g & \lambda_{27} & -G_6 & \bar{Y}'_g & -A_b & A_r & 0 \\
 & & & W_R^+ & \bar{Y}_b & -A'_g & 0 & A'_r & G_2 & -\bar{X}_b & -G_3 & \lambda_{28} & -\bar{Y}'_b & -A_g & 0 & A_r \\
-A'_r & 0 & A'_g & -A'_b & W_L^- & & & & Y'_r & 0 & -Y'_g & Y'_b & \lambda_{29} & -\bar{X}_r & -\bar{X}_b & -\bar{X}_g \\
0 & -A'_r & \bar{Y}_g & \bar{Y}_b & & W_L^- & & & 0 & Y'_r & \bar{A}_b & \bar{A}_g & -X_r & \lambda_{30} & -G_2 & -G_1 \\
-\bar{Y}_b & -A'_b & -\bar{Y}_r & 0 & & & W_L^- & & \bar{A}_g & Y'_b & -\bar{A}_r & 0 & -X_b & -G_1 & \lambda_{31} & -G_6 \\
\bar{Y}_g & -A'_g & 0 & -\bar{Y}_r & & & & W_L^- & \bar{A}_b & Y'_g & 0 & -\bar{A}_r & -X_g & -G_4 & -G_3 & \lambda_{32}
\end{bmatrix}_R
$$

The diagonal terms in the first block are explicitly

$$
\begin{aligned}
\lambda_1 &= +\frac{2\,G_7}{\sqrt{6}} + \frac{X_{B-L}}{2\sqrt{3}} + \frac{W_L^3}{\sqrt{2}} & \lambda_9 &= +\frac{2\,G_7}{\sqrt{6}} + \frac{X_{B-L}}{2\sqrt{3}} - \frac{W_L^3}{\sqrt{2}} \\
\lambda_2 &= -\frac{\sqrt{3}X_{B-L}}{2} + \frac{W_L^3}{\sqrt{2}} & \lambda_{10} &= -\frac{\sqrt{3}X_{B-L}}{2} - \frac{W_L^3}{\sqrt{2}} \\
\lambda_3 &= +\frac{2\,G_8}{\sqrt{6}} + \frac{X_{B-L}}{2\sqrt{3}} + \frac{W_L^3}{\sqrt{2}} & \lambda_{11} &= +\frac{2\,G_8}{\sqrt{6}} + \frac{X_{B-L}}{2\sqrt{3}} - \frac{W_L^3}{\sqrt{2}} \\
\lambda_4 &= -\frac{2\,(G_7+G_8)}{\sqrt{6}} + \frac{X_{B-L}}{2\sqrt{3}} + \frac{W_L^3}{\sqrt{2}} & \lambda_{12} &= -\frac{2\,(G_7+G_8)}{\sqrt{6}} + \frac{X_{B-L}}{2\sqrt{3}} - \frac{W_L^3}{\sqrt{2}} \\
\lambda_5 &= +\frac{\sqrt{3}X_{B-L}}{2} - \frac{W_R^3}{\sqrt{2}} & \lambda_{13} &= +\frac{\sqrt{3}X_{B-L}}{2} + \frac{W_R^3}{\sqrt{2}} \\
\lambda_6 &= -\frac{2\,G_7}{\sqrt{6}} - \frac{X_{B-L}}{2\sqrt{3}} - \frac{W_R^3}{\sqrt{2}} & \lambda_{14} &= -\frac{2\,G_7}{\sqrt{6}} - \frac{X_{B-L}}{2\sqrt{3}} + \frac{W_R^3}{\sqrt{2}} \\
\lambda_7 &= +\frac{2\,(G_7+G_8)}{\sqrt{6}} - \frac{X_{B-L}}{2\sqrt{3}} - \frac{W_R^3}{\sqrt{2}} & \lambda_{15} &= +\frac{2\,(G_7+G_8)}{\sqrt{6}} - \frac{X_{B-L}}{2\sqrt{3}} + \frac{W_R^3}{\sqrt{2}} \\
\lambda_8 &= -\frac{2\,G_8}{\sqrt{6}} - \frac{X_{B-L}}{2\sqrt{3}} - \frac{W_R^3}{\sqrt{2}} & \lambda_{16} &= -\frac{2\,G_8}{\sqrt{6}} - \frac{X_{B-L}}{2\sqrt{3}} + \frac{W_R^3}{\sqrt{2}}
\end{aligned}
\tag{4.3}
$$

The diagonal terms in the last block are explicitly

$$
\begin{aligned}
\lambda_{17} &= +\frac{2\,G_7}{\sqrt{6}} + \frac{X_{B-L}}{2\sqrt{3}} + \frac{W_R^3}{\sqrt{2}} & \lambda_{25} &= +\frac{2\,G_7}{\sqrt{6}} + \frac{X_{B-L}}{2\sqrt{3}} - \frac{W_R^3}{\sqrt{2}} \\
\lambda_{18} &= -\frac{\sqrt{3}X_{B-L}}{2} + \frac{W_R^3}{\sqrt{2}} & \lambda_{26} &= -\frac{\sqrt{3}X_{B-L}}{2} - \frac{W_R^3}{\sqrt{2}} \\
\lambda_{19} &= +\frac{2\,G_8}{\sqrt{6}} + \frac{X_{B-L}}{2\sqrt{3}} + \frac{W_R^3}{\sqrt{2}} & \lambda_{27} &= +\frac{2\,G_8}{\sqrt{6}} + \frac{X_{B-L}}{2\sqrt{3}} - \frac{W_R^3}{\sqrt{2}} \\
\lambda_{20} &= -\frac{2\,(G_7+G_8)}{\sqrt{6}} + \frac{X_{B-L}}{2\sqrt{3}} + \frac{W_R^3}{\sqrt{2}} & \lambda_{28} &= -\frac{2\,(G_7+G_8)}{\sqrt{6}} + \frac{X_{B-L}}{2\sqrt{3}} - \frac{W_R^3}{\sqrt{2}} \\
\lambda_{21} &= +\frac{\sqrt{3}X_{B-L}}{2} - \frac{W_L^3}{\sqrt{2}} & \lambda_{29} &= +\frac{\sqrt{3}X_{B-L}}{2} + \frac{W_L^3}{\sqrt{2}} \\
\lambda_{22} &= -\frac{2\,G_7}{\sqrt{6}} - \frac{X_{B-L}}{2\sqrt{3}} - \frac{W_L^3}{\sqrt{2}} & \lambda_{30} &= -\frac{2\,G_7}{\sqrt{6}} - \frac{X_{B-L}}{2\sqrt{3}} + \frac{W_L^3}{\sqrt{2}} \\
\lambda_{23} &= +\frac{2\,(G_7+G_8)}{\sqrt{6}} - \frac{X_{B-L}}{2\sqrt{3}} - \frac{W_L^3}{\sqrt{2}} & \lambda_{31} &= +\frac{2\,(G_7+G_8)}{\sqrt{6}} - \frac{X_{B-L}}{2\sqrt{3}} + \frac{W_L^3}{\sqrt{2}} \\
\lambda_{24} &= -\frac{2\,G_8}{\sqrt{6}} - \frac{X_{B-L}}{2\sqrt{3}} - \frac{W_L^3}{\sqrt{2}} & \lambda_{32} &= -\frac{2\,G_8}{\sqrt{6}} - \frac{X_{B-L}}{2\sqrt{3}} + \frac{W_L^3}{\sqrt{2}}
\end{aligned}
\tag{4.4}
$$

4.2 Basis B

The matrices $\Lambda_{11}, \Lambda_{12}, \Lambda_{21}$ and Λ_{22} are given respectively as below. It is seen that the gauge term is spread over a 32×32 matrix. The spinor that this gauge term acts on can be found through the eigenvalue operators in eq. (3.3) and eq. (3.9). We have

$$\Psi = \begin{bmatrix} u_r \\ e^c \\ d_b^c \\ u_g \\ d_g^c \\ u_b \\ \nu \\ d_r^c \\ u_r \\ e^c \\ d_b^c \\ u_g \\ d_g^c \\ u_b \\ \nu \\ d_r^c \\ d_r \\ -\nu^c \\ -u_b^c \\ d_g \\ -u_g^c \\ d_b \\ e \\ -u_r^c \\ d_r \\ -\nu^c \\ -u_b^c \\ d_g \\ -u_g^c \\ d_b \\ e \\ -u_r^c \end{bmatrix} \qquad \Psi_L = \begin{bmatrix} u_r \\ 0 \\ 0 \\ u_g \\ 0 \\ u_b \\ \nu \\ 0 \\ 0 \\ e^c \\ d_b^c \\ 0 \\ d_g^c \\ 0 \\ 0 \\ d_r^c \\ 0 \\ -\nu^c \\ -u_b^c \\ 0 \\ -u_g^c \\ 0 \\ 0 \\ -u_r^c \\ d_r \\ 0 \\ 0 \\ d_g \\ 0 \\ d_b \\ e \\ 0 \end{bmatrix}_L \qquad \Psi_R = \begin{bmatrix} 0 \\ e^c \\ d_b^c \\ 0 \\ d_g^c \\ 0 \\ 0 \\ d_r^c \\ u_r \\ 0 \\ 0 \\ u_g \\ 0 \\ u_b \\ \nu \\ 0 \\ d_r \\ 0 \\ 0 \\ d_g \\ 0 \\ d_b \\ e \\ 0 \\ 0 \\ -\nu^c \\ -u_b^c \\ 0 \\ -u_g^c \\ 0 \\ 0 \\ -u_r^c \end{bmatrix}_R \tag{4.5}$$

Herein the first one is the total spinor and is $\Psi = \Psi_L + \Psi_R$ and the other chiral components are reduced through Γ_{five}. We have

$$\Psi_L = \frac{1}{2}\left(1 + \Gamma_{five}\right)\ \Psi \tag{4.6}$$

$$\Psi_R = \frac{1}{2}\left(1 - \Gamma_{five}\right)\ \Psi \tag{4.7}$$

The chiral components have the following feature that every single fermion subject to the Dirac equation in Ψ_L is understood to be a left handed spinor and every single fermion in Ψ_R is understood to be a right handed spinor, so that the index L and R perform a double duty both for the Dirac representation of a single fermion and the $SO(10)$ representation of a chiral family spinor.

$$
\begin{bmatrix}
\lambda_1 & . & . & G_4 & . & -G_5 & \bar{X}_r & . & . & \bar{Y'}_r & -A'_g & . & A'_b & . & . & . \\
. & \lambda_2 & -\bar{X}_b & . & -\bar{X}_g & . & . & \bar{X}_r & -A_r & . & . & -A'_g & . & A'_b & . & . \\
. & -X_b & \lambda_3 & . & -G_6 & . & . & G_5 & -\bar{Y}_g & . & . & -\bar{Y'}_r & . & . & A'_b & . \\
G_1 & . & . & \lambda_4 & . & -G_6 & \bar{X}_g & . & . & -\bar{Y}_g & A_r & . & . & . & . & A'_b \\
. & -X_g & -G_3 & . & \lambda_5 & . & . & G_4 & \bar{Y}_b & . & . & . & . & \bar{Y'}_r & A'_g & . \\
-G_2 & . & . & -G_3 & . & \lambda_6 & -\bar{X}_b & . & . & \bar{Y}_b & . & . & A_r & . & . & A'_g \\
X_r & . & . & X_g & . & -X_b & \lambda_7 & . & . & . & \bar{Y}_b & . & \bar{Y}_g & . & . & \bar{Y'}_r \\
. & X_r & G_2 & . & G_1 & . & . & \lambda_8 & . & . & . & \bar{Y}_b & . & \bar{Y}_g & -A_r & . \\
. & \bar{A}'_r & Y_g & . & -Y_b & . & . & . & \lambda_9 & . & . & G_4 & . & -G_5 & \bar{X}_r & . \\
Y'_r & . & . & Y_g & . & -Y_b & . & . & . & \lambda_{10} & -\bar{X}_b & . & -\bar{X}_g & . & . & \bar{X}_r \\
\bar{A}'_g & . & . & -\bar{A}'_r & . & . & -Y_b & . & . & -X_b & \lambda_{11} & . & -G_6 & . & . & G_5 \\
. & \bar{A}'_g & -Y'_r & . & . & . & . & -Y_b & G_1 & . & . & \lambda_{12} & . & -G_6 & \bar{X}_g & . \\
-\bar{A}'_b & . & . & . & . & -\bar{A}'_r & -Y_g & . & . & -X_g & -G_3 & . & \lambda_{13} & . & . & G_4 \\
. & -\bar{A}'_b & . & . & -Y'_r & . & . & -Y_g & -G_2 & . & . & -G_3 & . & \lambda_{14} & -\bar{X}_b & . \\
. & . & -\bar{A}'_b & . & -\bar{A}'_g & . & . & \bar{A}'_r & X_r & . & . & \bar{X}_g & . & -\bar{X}_b & \lambda_{15} & . \\
. & . & . & -\bar{A}'_b & . & -\bar{A}'_g & Y'_r & . & . & X_r & G_2 & . & G_1 & . & . & \lambda_{16}
\end{bmatrix}
$$

$$
\begin{bmatrix}
. & -\bar{Y}'_r & A_g & . & -A_b & . & . & . & W_R^+ & . & . & . & . & . & . & . \\
A_r & . & . & A_g & . & -A_b & . & . & . & W_R^+ & . & . & . & . & . & . \\
-\bar{Y}'_g & . & . & \bar{Y}'_r & . & . & -A_b & . & . & . & W_R^+ & . & . & . & . & . \\
. & -\bar{Y}'_g & -A_r & . & . & . & . & -A_b & . & . & . & W_R^+ & . & . & . & . \\
\bar{Y}'_b & . & . & . & . & \bar{Y}'_r & -A_g & . & . & . & . & . & W_R^+ & . & . & . \\
. & \bar{Y}'_b & . & . & -A_r & . & . & -A_g & . & . & . & . & . & W_R^+ & . & . \\
. & . & \bar{Y}'_b & . & \bar{Y}'_g & . & . & -\bar{Y}'_r & . & . & . & . & . & . & W_R^+ & . \\
. & . & . & \bar{Y}'_b & . & \bar{Y}'_g & A_r & . & . & . & . & . & . & . & . & W_R^+ \\
-W_L^+ & . & . & . & . & . & . & . & . & \bar{Y}'_r & -A_g & . & A_b & . & . & . \\
. & -W_L^+ & . & . & . & . & . & . & -A_r & . & . & -A_g & . & A_b & . & . \\
. & . & -W_L^+ & . & . & . & . & . & \bar{Y}'_g & . & . & -\bar{Y}'_r & . & . & A_b & . \\
. & . & . & -W_L^+ & . & . & . & . & . & \bar{Y}'_g & A_r & . & . & . & . & A_b \\
. & . & . & . & -W_L^+ & . & . & . & -\bar{Y}'_b & . & . & . & . & -\bar{Y}'_r & A_g & . \\
. & . & . & . & . & -W_L^+ & . & . & . & -\bar{Y}'_b & . & . & A_r & . & . & A_g \\
. & . & . & . & . & . & -W_L^+ & . & . & . & -\bar{Y}'_b & . & -\bar{Y}'_g & . & . & \bar{Y}'_r \\
. & . & . & . & . & . & . & -W_L^+ & . & . & . & -\bar{Y}'_b & . & -\bar{Y}'_g & -A_r & .
\end{bmatrix}
$$

$$\begin{bmatrix}
\cdot & -\bar{A}_r & Y'_g & \cdot & -Y'_b & \cdot & \cdot & \cdot & -W_L^- & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
Y'_r & \cdot & \cdot & Y'_g & \cdot & -Y'_b & \cdot & \cdot & \cdot & -W_L^- & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
-\bar{A}_g & \cdot & \cdot & \bar{A}_r & \cdot & \cdot & -Y'_b & \cdot & \cdot & \cdot & -W_L^- & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & -\bar{A}_g & -Y'_r & \cdot & \cdot & \cdot & \cdot & -Y'_b & \cdot & \cdot & \cdot & -W_L^- & \cdot & \cdot & \cdot & \cdot \\
\bar{A}_b & \cdot & \cdot & \cdot & \cdot & \bar{A}_r & -Y'_g & \cdot & \cdot & \cdot & \cdot & \cdot & -W_L^- & \cdot & \cdot & \cdot \\
\cdot & \bar{A}_b & \cdot & \cdot & -Y'_r & \cdot & \cdot & -Y'_g & \cdot & \cdot & \cdot & \cdot & \cdot & -W_L^- & \cdot & \cdot \\
\cdot & \cdot & \bar{A}_b & \cdot & \bar{A}_g & \cdot & \cdot & -\bar{A}_r & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -W_L^- & \cdot \\
\cdot & \cdot & \cdot & \bar{A}_b & \cdot & \bar{A}_g & Y'_r & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -W_L^- \\
W_R^- & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \bar{A}_r & -Y'_g & \cdot & Y'_b & \cdot & \cdot & \cdot \\
\cdot & W_R^- & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -Y'_r & \cdot & \cdot & -Y'_g & \cdot & Y'_b & \cdot & \cdot \\
\cdot & \cdot & W_R^- & \cdot & \cdot & \cdot & \cdot & \cdot & \bar{A}_g & \cdot & \cdot & -\bar{A}_r & \cdot & \cdot & Y'_b & \cdot \\
\cdot & \cdot & \cdot & W_R^- & \cdot & \cdot & \cdot & \cdot & \cdot & \bar{A}_g & Y'_r & \cdot & \cdot & \cdot & \cdot & Y'_b \\
\cdot & \cdot & \cdot & \cdot & W_R^- & \cdot & \cdot & \cdot & -\bar{A}_b & \cdot & \cdot & \cdot & \cdot & -\bar{A}_r & Y'_g & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & W_R^- & \cdot & \cdot & \cdot & -\bar{A}_b & \cdot & \cdot & Y'_r & \cdot & \cdot & Y'_g \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & W_R^- & \cdot & \cdot & \cdot & -\bar{A}_b & \cdot & -\bar{A}_g & \cdot & \cdot & \bar{A}_r \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & W_R^- & \cdot & \cdot & \cdot & -\bar{A}_b & \cdot & -\bar{A}_g & -Y'_r & \cdot
\end{bmatrix}$$

$$\begin{bmatrix}
\lambda_{17} & \cdot & \cdot & G_4 & \cdot & -G_5 & \bar{X}_r & \cdot & \cdot & \bar{Y'}_r & -A'_g & \cdot & A'_b & \cdot & \cdot & \cdot \\
\cdot & \lambda_{18} & -\bar{X}_b & \cdot & -\bar{X}_g & \cdot & \cdot & \bar{X}_r & -A_r & \cdot & \cdot & -A'_g & \cdot & A'_b & \cdot & \cdot \\
\cdot & -X_b & \lambda_{19} & \cdot & -G_6 & \cdot & \cdot & G_5 & -\bar{Y}_g & \cdot & \cdot & -\bar{Y'}_r & \cdot & \cdot & A'_b & \cdot \\
G_1 & \cdot & \cdot & \lambda_{20} & \cdot & -G_6 & \bar{X}_g & \cdot & \cdot & -\bar{Y}_g & A_r & \cdot & \cdot & \cdot & \cdot & A'_b \\
\cdot & -X_g & -G_3 & \cdot & \lambda_{21} & \cdot & \cdot & G_4 & \bar{Y}_b & \cdot & \cdot & \cdot & \cdot & \bar{Y'}_r & A'_g & \cdot \\
-G_2 & \cdot & \cdot & -G_3 & \cdot & \lambda_{22} & -\bar{X}_b & \cdot & \cdot & \bar{Y}_b & \cdot & \cdot & A_r & \cdot & \cdot & A'_g \\
X_r & \cdot & \cdot & X_g & \cdot & -X_b & \lambda_{23} & \cdot & \cdot & \cdot & \bar{Y}_b & \cdot & \bar{Y}_g & \cdot & \cdot & \bar{Y'}_r \\
\cdot & X_r & G_2 & \cdot & G_1 & \cdot & \cdot & \lambda_{24} & \cdot & \cdot & \cdot & \bar{Y}_b & \cdot & \bar{Y}_g & -A_r & \cdot \\
\cdot & \bar{A}'_r & Y_g & \cdot & -Y_b & \cdot & \cdot & \cdot & \lambda_{25} & \cdot & \cdot & G_4 & \cdot & -G_5 & \bar{X}_r & \cdot \\
Y'_r & \cdot & \cdot & Y_g & \cdot & -Y_b & \cdot & \cdot & \cdot & \lambda_{26} & -\bar{X}_b & \cdot & -\bar{X}_g & \cdot & \cdot & \bar{X}_r \\
\bar{A}'_g & \cdot & \cdot & -\bar{A}'_r & \cdot & \cdot & -Y_b & \cdot & \cdot & -X_b & \lambda_{27} & \cdot & -G_6 & \cdot & \cdot & G_5 \\
\cdot & \bar{A}'_g & -Y'_r & \cdot & \cdot & \cdot & \cdot & -Y_b & G_1 & \cdot & \cdot & \lambda_{28} & \cdot & -G_6 & \bar{X}_g & \cdot \\
-\bar{A}'_b & \cdot & \cdot & \cdot & \cdot & -\bar{A}'_r & -Y_g & \cdot & \cdot & -X_g & -G_3 & \cdot & \lambda_{29} & \cdot & \cdot & G_4 \\
\cdot & -\bar{A}'_b & \cdot & \cdot & -Y'_r & \cdot & \cdot & -Y_g & -G_2 & \cdot & \cdot & -G_3 & \cdot & \lambda_{30} & -\bar{X}_b & \cdot \\
\cdot & \cdot & -\bar{A}'_b & \cdot & -\bar{A}'_g & \cdot & \cdot & \bar{A}'_r & X_r & \cdot & \cdot & X_g & \cdot & -X_b & \lambda_{31} & \cdot \\
\cdot & \cdot & \cdot & -\bar{A}'_b & \cdot & -\bar{A}'_g & Y'_r & \cdot & \cdot & X_r & G_2 & \cdot & G_1 & \cdot & \cdot & \lambda_{32}
\end{bmatrix}$$

The diagonal entries are found as

$$\begin{aligned}
\lambda_1 &= +\frac{2\,G_7}{\sqrt{6}} + \frac{X_{B-L}}{2\sqrt{3}} + \frac{W_L^3}{\sqrt{2}} & \lambda_9 &= +\frac{2\,G_7}{\sqrt{6}} + \frac{X_{B-L}}{2\sqrt{3}} + \frac{W_R^3}{\sqrt{2}} \\
\lambda_2 &= +\frac{\sqrt{3}X_{B-L}}{2} + \frac{W_L^3}{\sqrt{2}} & \lambda_{10} &= +\frac{\sqrt{3}X_{B-L}}{2} + \frac{W_R^3}{\sqrt{2}} \\
\lambda_3 &= +\frac{2\,(G_7+G_8)}{\sqrt{6}} - \frac{X_{B-L}}{2\sqrt{3}} + \frac{W_L^3}{\sqrt{2}} & \lambda_{11} &= +\frac{2\,(G_7+G_8)}{\sqrt{6}} - \frac{X_{B-L}}{2\sqrt{3}} + \frac{W_R^3}{\sqrt{2}} \\
\lambda_4 &= +\frac{2\,G_8}{\sqrt{6}} + \frac{X_{B-L}}{2\sqrt{3}} + \frac{W_L^3}{\sqrt{2}} & \lambda_{12} &= +\frac{2\,G_8}{\sqrt{6}} + \frac{X_{B-L}}{2\sqrt{3}} + \frac{W_R^3}{\sqrt{2}} \\
\lambda_5 &= -\frac{2\,G_8}{\sqrt{6}} - \frac{X_{B-L}}{2\sqrt{3}} + \frac{W_L^3}{\sqrt{2}} & \lambda_{13} &= -\frac{2\,G_8}{\sqrt{6}} - \frac{X_{B-L}}{2\sqrt{3}} + \frac{W_R^3}{\sqrt{2}} \\
\lambda_6 &= -\frac{2\,(G_7+G_8)}{\sqrt{6}} + \frac{X_{B-L}}{2\sqrt{3}} + \frac{W_L^3}{\sqrt{2}} & \lambda_{14} &= -\frac{2\,(G_7+G_8)}{\sqrt{6}} + \frac{X_{B-L}}{2\sqrt{3}} + \frac{W_R^3}{\sqrt{2}} \\
\lambda_7 &= -\frac{\sqrt{3}X_{B-L}}{2} + \frac{W_L^3}{\sqrt{2}} & \lambda_{15} &= -\frac{\sqrt{3}X_{B-L}}{2} + \frac{W_R^3}{\sqrt{2}} \\
\lambda_8 &= -\frac{2\,G_7}{\sqrt{6}} - \frac{X_{B-L}}{2\sqrt{3}} + \frac{W_L^3}{\sqrt{2}} & \lambda_{16} &= -\frac{2\,G_7}{\sqrt{6}} - \frac{X_{B-L}}{2\sqrt{3}} + \frac{W_R^3}{\sqrt{2}}
\end{aligned} \tag{4.8}$$

$$\begin{aligned}
\lambda_{17} &= +\frac{2\,G_7}{\sqrt{6}} + \frac{X_{B-L}}{2\sqrt{3}} - \frac{W_R^3}{\sqrt{2}} & \lambda_{25} &= +\frac{2\,G_7}{\sqrt{6}} + \frac{X_{B-L}}{2\sqrt{3}} - \frac{W_L^3}{\sqrt{2}} \\
\lambda_{18} &= +\frac{\sqrt{3}X_{B-L}}{2} - \frac{W_R^3}{\sqrt{2}} & \lambda_{26} &= +\frac{\sqrt{3}X_{B-L}}{2} - \frac{W_L^3}{\sqrt{2}} \\
\lambda_{19} &= +\frac{2\,(G_7+G_8)}{\sqrt{6}} - \frac{X_{B-L}}{2\sqrt{3}} - \frac{W_R^3}{\sqrt{2}} & \lambda_{27} &= +\frac{2\,(G_7+G_8)}{\sqrt{6}} - \frac{X_{B-L}}{2\sqrt{3}} - \frac{W_L^3}{\sqrt{2}} \\
\lambda_{20} &= +\frac{2\,G_8}{\sqrt{6}} + \frac{X_{B-L}}{2\sqrt{3}} - \frac{W_R^3}{\sqrt{2}} & \lambda_{28} &= +\frac{2\,G_8}{\sqrt{6}} + \frac{X_{B-L}}{2\sqrt{3}} - \frac{W_L^3}{\sqrt{2}} \\
\lambda_{21} &= -\frac{2\,G_8}{\sqrt{6}} - \frac{X_{B-L}}{2\sqrt{3}} - \frac{W_R^3}{\sqrt{2}} & \lambda_{29} &= -\frac{2\,G_8}{\sqrt{6}} - \frac{X_{B-L}}{2\sqrt{3}} - \frac{W_L^3}{\sqrt{2}} \\
\lambda_{22} &= -\frac{2\,(G_7+G_8)}{\sqrt{6}} + \frac{X_{B-L}}{2\sqrt{3}} - \frac{W_R^3}{\sqrt{2}} & \lambda_{30} &= -\frac{2\,(G_7+G_8)}{\sqrt{6}} + \frac{X_{B-L}}{2\sqrt{3}} - \frac{W_L^3}{\sqrt{2}} \\
\lambda_{23} &= -\frac{\sqrt{3}X_{B-L}}{2} - \frac{W_R^3}{\sqrt{2}} & \lambda_{31} &= -\frac{\sqrt{3}X_{B-L}}{2} - \frac{W_L^3}{\sqrt{2}} \\
\lambda_{24} &= -\frac{2\,G_7}{\sqrt{6}} - \frac{X_{B-L}}{2\sqrt{3}} - \frac{W_R^3}{\sqrt{2}} & \lambda_{32} &= -\frac{2\,G_7}{\sqrt{6}} - \frac{X_{B-L}}{2\sqrt{3}} - \frac{W_L^3}{\sqrt{2}}
\end{aligned} \tag{4.9}$$

4.3 Basis C

For this basis, Λ_{12} and Λ_{21} are blocks of zeros and the below matrices are respectively the Λ_{11} and Λ_{22} parts of the gauge term in eq. (4.1).

$$
\begin{bmatrix}
\lambda_1 & G_4 & G_5 & \bar{X}_r & W_L^- & & & & 0 & -A'_b & A'_g & \bar{Y}_r & 0 & Y'_b & -Y'_g & \bar{A}_r \\
G_1 & \lambda_2 & G_6 & \bar{X}_g & & W_L^- & & & A'_b & 0 & -A'_r & \bar{Y}_g & -Y'_b & 0 & Y'_r & \bar{A}_g \\
G_2 & G_3 & \lambda_3 & \bar{X}_b & & & W_L^- & & -A'_g & A'_r & 0 & \bar{Y}_b & Y'_g & -Y'_r & 0 & \bar{A}_b \\
X_r & X_g & X_b & \lambda_4 & & & & W_L^- & -\bar{Y}_r & -\bar{Y}_g & -\bar{Y}_b & 0 & -\bar{A}_r & -\bar{A}_g & -\bar{A}_b & 0 \\
W_L^+ & & & & \lambda_5 & G_4 & G_5 & \bar{X}_r & 0 & A_b & -A_g & \bar{Y}'_r & 0 & -Y_b & Y_g & \bar{A}'_r \\
& W_L^+ & & & G_1 & \lambda_6 & G_6 & \bar{X}_g & -A_b & 0 & A_r & \bar{Y}'_g & Y_b & 0 & -Y_r & \bar{A}'_g \\
& & W_L^+ & & G_2 & G_3 & \lambda_7 & \bar{X}_b & A_g & -A_r & 0 & \bar{Y}'_b & -Y_g & Y_r & 0 & \bar{A}'_b \\
& & & W_L^+ & X_r & X_g & X_b & \lambda_8 & -\bar{Y}'_r & -\bar{Y}'_g & -\bar{Y}'_b & 0 & -\bar{A}'_r & -\bar{A}'_g & -\bar{A}'_b & 0 \\
0 & \bar{A}'_b & -\bar{A}'_g & -Y_r & 0 & -\bar{A}_b & \bar{A}_g & -Y'_r & \lambda_9 & -G_1 & -G_2 & -X_r & W_R^- & & & \\
-\bar{A}'_b & 0 & \bar{A}'_r & -Y_g & \bar{A}_b & 0 & -\bar{A}_r & -Y'_g & -G_4 & \lambda_{10} & -G_3 & -X_g & & W_R^- & & \\
\bar{A}'_g & -\bar{A}'_r & 0 & -Y_b & -\bar{A}_g & \bar{A}_r & 0 & -Y'_b & -G_5 & -G_6 & \lambda_{11} & -X_b & & & W_R^- & \\
Y_r & Y_g & Y_b & 0 & Y'_r & Y'_g & Y'_b & 0 & -\bar{X}_r & -\bar{X}_g & -\bar{X}_b & \lambda_{12} & & & & W_R^- \\
0 & -\bar{Y}'_b & \bar{Y}'_g & -A_r & 0 & \bar{Y}_b & -\bar{Y}_g & -A'_r & W_R^+ & & & & \lambda_{13} & -G_1 & -G_2 & -X_r \\
\bar{Y}'_b & 0 & -\bar{Y}'_r & -A_g & -\bar{Y}_b & 0 & \bar{Y}_r & -A'_g & & W_R^+ & & & -G_4 & \lambda_{14} & -G_3 & -X_g \\
-\bar{Y}'_g & \bar{Y}'_r & 0 & -A_b & \bar{Y}_g & -\bar{Y}_r & 0 & -A'_b & & & W_R^+ & & -G_5 & -G_6 & \lambda_{15} & -X_b \\
A_r & A_g & A_b & 0 & A'_r & A'_g & A'_b & 0 & & & & W_R^+ & -\bar{X}_r & -\bar{X}_g & -\bar{X}_b & \lambda_{16}
\end{bmatrix}_L
$$

$$
\begin{bmatrix}
\lambda_{17} & G_4 & G_5 & \bar{X}_r & W_R^- & & & & 0 & Y_b & -Y_g & -\bar{A}'_r & 0 & Y'_b & -Y'_g & \bar{A}_r \\
G_1 & \lambda_{18} & G_6 & \bar{X}_g & & W_R^- & & & -Y_b & 0 & Y_r & -\bar{A}'_g & -Y'_b & 0 & Y'_r & \bar{A}_g \\
G_2 & G_3 & \lambda_{19} & \bar{X}_b & & & W_R^- & & Y_g & -Y_r & 0 & -\bar{A}'_b & Y'_g & -Y'_r & 0 & \bar{A}_b \\
X_r & X_g & X_b & \lambda_{20} & & & & W_R^- & \bar{A}'_r & \bar{A}'_g & \bar{A}'_b & 0 & -\bar{A}_r & -\bar{A}_g & -\bar{A}_b & 0 \\
W_R^+ & & & & \lambda_{21} & G_4 & G_5 & \bar{X}_r & 0 & A_b & -A_g & \bar{Y}'_r & 0 & A'_b & -A'_g & -\bar{Y}_r \\
& W_R^+ & & & G_1 & \lambda_{22} & G_6 & \bar{X}_g & -A_b & 0 & A_r & \bar{Y}'_g & -A'_b & 0 & A'_r & -\bar{Y}_g \\
& & W_R^+ & & G_2 & G_3 & \lambda_{23} & \bar{X}_b & A_g & -A_r & 0 & \bar{Y}'_b & A'_g & -A'_r & 0 & -\bar{Y}_b \\
& & & W_R^+ & X_r & X_g & X_b & \lambda_{24} & -\bar{Y}'_r & -\bar{Y}'_g & -\bar{Y}'_b & 0 & \bar{Y}_r & \bar{Y}_g & \bar{Y}_b & 0 \\
0 & -\bar{Y}_b & \bar{Y}_g & A'_r & 0 & -\bar{A}_b & \bar{A}_g & -Y'_r & \lambda_{25} & -G_1 & -G_2 & -X_r & W_L^- & & & \\
\bar{Y}_b & 0 & -\bar{Y}_r & A'_g & \bar{A}_b & 0 & -\bar{A}_r & -Y'_g & -G_4 & \lambda_{26} & -G_3 & -X_g & & W_L^- & & \\
-\bar{Y}_g & \bar{Y}_r & 0 & A'_b & -\bar{A}_g & \bar{A}_r & 0 & -Y'_b & -G_5 & -G_6 & \lambda_{27} & -X_b & & & W_L^- & \\
-A'_r & -A'_g & -A'_b & 0 & Y'_r & Y'_g & Y'_b & 0 & -\bar{X}_r & -\bar{X}_g & -\bar{X}_b & \lambda_{28} & & & & W_L^- \\
0 & -\bar{Y}'_b & \bar{Y}'_g & -A_r & 0 & -\bar{A}'_b & \bar{A}'_g & Y_r & W_L^+ & & & & \lambda_{29} & -G_1 & -G_2 & -X_r \\
\bar{Y}'_b & 0 & -\bar{Y}'_r & -A_g & \bar{A}'_b & 0 & -\bar{A}'_r & Y_g & & W_L^+ & & & -G_4 & \lambda_{30} & -G_3 & -X_g \\
-\bar{Y}'_g & \bar{Y}'_r & 0 & -A_b & -\bar{A}'_g & \bar{A}'_r & 0 & Y_b & & & W_L^+ & & -G_5 & -G_6 & \lambda_{31} & -X_b \\
A_r & A_g & A_b & 0 & -Y_r & -Y_g & -Y_b & 0 & & & & W_L^+ & -\bar{X}_r & -\bar{X}_g & -\bar{X}_b & \lambda_{32}
\end{bmatrix}_R
$$

The empty spots again correspond to single zero entries and are left blank for a better appearance. The resulting fermion assignment of the spinor in basis C determined by the eigenvalue operators in eq. (3.3) and eq. (3.9) turn out to be

$$\Psi_L = \begin{bmatrix} u_r \\ u_g \\ u_b \\ \nu \\ d_r \\ d_g \\ d_b \\ e \\ d_r^c \\ d_g^c \\ d_b^c \\ e^c \\ -u_r^c \\ -u_g^c \\ -u_b^c \\ -\nu^c \end{bmatrix}_L \quad , \quad \Psi_R = \begin{bmatrix} u_r \\ u_g \\ u_b \\ \nu \\ d_r \\ d_g \\ d_b \\ e \\ d_r^c \\ d_g^c \\ d_b^c \\ e^c \\ -u_r^c \\ -u_g^c \\ -u_b^c \\ -\nu^c \end{bmatrix}_R \quad , \quad \Psi = \begin{pmatrix} \Psi_L \\ \Psi_R \end{pmatrix} \tag{4.10}$$

where Ψ is the total spinor. It should be noted that the single fermion entries in Ψ_L and Ψ_R are understood to be left and right handed spinors respectively. The diagonal terms are explicitly

$$\begin{aligned}
\lambda_1 &= +\frac{2\,G_7}{\sqrt{6}} + \frac{X_{B-L}}{2\sqrt{3}} + \frac{W_L^3}{\sqrt{2}} & \lambda_9 &= -\frac{2\,G_7}{\sqrt{6}} - \frac{X_{B-L}}{2\sqrt{3}} + \frac{W_R^3}{\sqrt{2}} \\
\lambda_2 &= +\frac{2\,G_8}{\sqrt{6}} + \frac{X_{B-L}}{2\sqrt{3}} + \frac{W_L^3}{\sqrt{2}} & \lambda_{10} &= -\frac{2\,G_8}{\sqrt{6}} - \frac{X_{B-L}}{2\sqrt{3}} + \frac{W_R^3}{\sqrt{2}} \\
\lambda_3 &= -\frac{2\,(G_7+G_8)}{\sqrt{6}} + \frac{X_{B-L}}{2\sqrt{3}} + \frac{W_L^3}{\sqrt{2}} & \lambda_{11} &= +\frac{2\,(G_7+G_8)}{\sqrt{6}} - \frac{X_{B-L}}{2\sqrt{3}} + \frac{W_R^3}{\sqrt{2}} \\
\lambda_4 &= -\frac{\sqrt{3}}{2}X_{B-L} + \frac{W_L^3}{\sqrt{2}} & \lambda_{12} &= +\frac{\sqrt{3}}{2}X_{B-L} + \frac{W_R^3}{\sqrt{2}} \\
\lambda_5 &= +\frac{2\,G_7}{\sqrt{6}} + \frac{X_{B-L}}{2\sqrt{3}} - \frac{W_L^3}{\sqrt{2}} & \lambda_{13} &= -\frac{2\,G_7}{\sqrt{6}} - \frac{X_{B-L}}{2\sqrt{3}} - \frac{W_R^3}{\sqrt{2}} \\
\lambda_6 &= +\frac{2\,G_8}{\sqrt{6}} + \frac{X_{B-L}}{2\sqrt{3}} - \frac{W_L^3}{\sqrt{2}} & \lambda_{14} &= -\frac{2\,G_8}{\sqrt{6}} - \frac{X_{B-L}}{2\sqrt{3}} - \frac{W_R^3}{\sqrt{2}} \\
\lambda_7 &= -\frac{2\,(G_7+G_8)}{\sqrt{6}} + \frac{X_{B-L}}{2\sqrt{3}} - \frac{W_L^3}{\sqrt{2}} & \lambda_{15} &= +\frac{2\,(G_7+G_8)}{\sqrt{6}} - \frac{X_{B-L}}{2\sqrt{3}} - \frac{W_R^3}{\sqrt{2}} \\
\lambda_8 &= -\frac{\sqrt{3}}{2}X_{B-L} - \frac{W_L^3}{\sqrt{2}} & \lambda_{16} &= +\frac{\sqrt{3}}{2}X_{B-L} - \frac{W_R^3}{\sqrt{2}}
\end{aligned} \tag{4.11}$$

$$\begin{aligned}
\lambda_{17} &= +\frac{2\,G_7}{\sqrt{6}} + \frac{X_{B-L}}{2\sqrt{3}} + \frac{W_R^3}{\sqrt{2}} & \lambda_{25} &= -\frac{2\,G_7}{\sqrt{6}} - \frac{X_{B-L}}{2\sqrt{3}} + \frac{W_L^3}{\sqrt{2}} \\
\lambda_{18} &= +\frac{2\,G_8}{\sqrt{6}} + \frac{X_{B-L}}{2\sqrt{3}} + \frac{W_R^3}{\sqrt{2}} & \lambda_{26} &= -\frac{2\,G_8}{\sqrt{6}} - \frac{X_{B-L}}{2\sqrt{3}} + \frac{W_L^3}{\sqrt{2}} \\
\lambda_{19} &= -\frac{2\,(G_7+G_8)}{\sqrt{6}} + \frac{X_{B-L}}{2\sqrt{3}} + \frac{W_R^3}{\sqrt{2}} & \lambda_{27} &= +\frac{2\,(G_7+G_8)}{\sqrt{6}} - \frac{X_{B-L}}{2\sqrt{3}} + \frac{W_L^3}{\sqrt{2}} \\
\lambda_{20} &= -\frac{\sqrt{3}}{2}X_{B-L} + \frac{W_R^3}{\sqrt{2}} & \lambda_{28} &= +\frac{\sqrt{3}}{2}X_{B-L} + \frac{W_L^3}{\sqrt{2}} \\
\lambda_{21} &= +\frac{2\,G_7}{\sqrt{6}} + \frac{X_{B-L}}{2\sqrt{3}} - \frac{W_R^3}{\sqrt{2}} & \lambda_{29} &= -\frac{2\,G_7}{\sqrt{6}} - \frac{X_{B-L}}{2\sqrt{3}} - \frac{W_L^3}{\sqrt{2}} \\
\lambda_{22} &= +\frac{2\,G_8}{\sqrt{6}} + \frac{X_{B-L}}{2\sqrt{3}} - \frac{W_R^3}{\sqrt{2}} & \lambda_{30} &= -\frac{2\,G_8}{\sqrt{6}} - \frac{X_{B-L}}{2\sqrt{3}} - \frac{W_L^3}{\sqrt{2}} \\
\lambda_{23} &= -\frac{2\,(G_7+G_8)}{\sqrt{6}} + \frac{X_{B-L}}{2\sqrt{3}} - \frac{W_R^3}{\sqrt{2}} & \lambda_{31} &= +\frac{2\,(G_7+G_8)}{\sqrt{6}} - \frac{X_{B-L}}{2\sqrt{3}} - \frac{W_L^3}{\sqrt{2}} \\
\lambda_{24} &= -\frac{\sqrt{3}}{2}X_{B-L} - \frac{W_R^3}{\sqrt{2}} & \lambda_{32} &= +\frac{\sqrt{3}}{2}X_{B-L} - \frac{W_L^3}{\sqrt{2}}
\end{aligned} \tag{4.12}$$

4.3.1 The Family Spinor: Decompositions of the 16

From the Lie algebra of $SO(10)$, we showed that $SO(6) \times SO(4)$ is a maximal subgroup of $SO(10)$. This was stated in eq. (2.28). The isomorphism between $SU(4)\times SU(2)_L\times SU(2)_R$ and this maximal subgroup was shown in § 2.5. From these considerations, it is possible to construct the generators of $SU(4) \times SU(2)_L \times SU(2)_R$. This helps us to elaborate the various decompositions of the fermion multiplet under $SU(4) \times SU(2)_L \times SU(2)_R$ and under its subgroups. The size of the matrix representation of $SU(4)\times SU(2)_L\times SU(2)_R$ is 32 which is determined from the clifford algebra in eq. (2.11). Since the generators of $SU(2)_L$ and $SU(2)_R$ should mutually commute we divide the 32×32 matrix into two 16×16 blocks and place along the diagonal the respective group generators as shown in Table (4.1) where I are unit matrices of 4×4 size. Indeed the above matrices are astonishingly

$$L_3 = \frac{1}{2}\begin{bmatrix} I & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -I & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -I \end{bmatrix}, \quad R_3 = \frac{1}{2}\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & I & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -I & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & I & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -I & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$L_1 = \frac{i}{2}\begin{bmatrix} 0 & -I & 0 & 0 & 0 & 0 & 0 & 0 \\ I & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -I \\ 0 & 0 & 0 & 0 & 0 & 0 & I & 0 \end{bmatrix}, \quad R_1 = \frac{i}{2}\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -I & 0 & 0 & 0 & 0 \\ 0 & 0 & I & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -I & 0 & 0 \\ 0 & 0 & 0 & 0 & I & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$L_2 = \frac{1}{2}\begin{bmatrix} 0 & I & 0 & 0 & 0 & 0 & 0 & 0 \\ I & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & I \\ 0 & 0 & 0 & 0 & 0 & 0 & I & 0 \end{bmatrix}, \quad R_2 = \frac{1}{2}\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & I & 0 & 0 & 0 & 0 \\ 0 & 0 & I & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & I & 0 & 0 \\ 0 & 0 & 0 & 0 & I & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Tab. 4.1: Matrix representations of the $SU(2)_L$ and $SU(2)_R$ groups in $SO(10)$

interesting while the size of the representation is equal to the number of states that one can build from a single fermion family. i.e., f_L + f^c_L + f_R + f^c_R produce 32 fields where f denotes fermions. The above representation dictates us to reserve the first 8 positions to f_L and the subsequent 8 positions for f^c_L. Similarly the third 8 positions and the last 8 positions should be reserved for f_R + f^c_R respectively. This is in perfect harmony with fact that the $SU(4)$ representation in its fundamental form are in 4×4 size. So one can place the $SU(4)$ representation along the diagonal respecting the above assignment. Since the I are unit matrices, all the $SU(4)$ generators will also commute with the $SU(2)_L \times SU(2)_R$ generators.

The appropriate form of the $SU(4)$ generators are given in Table (4.2) where the embedded $\mathbf{U}_i$ generators are given in Table (3.2). These 32×32 matrices are normalized to $Tr[(U_i)^2] = 4$ like the isospin generators and satisfy the $SU(4)$ Lie algebra in eq. (2.31) with the structure constants in Table (2.1). The fermion assignment that suits this reducibly constructed representation is summarized in Table (4.3). The left-right and up-down arrows, $\leftrightarrow, \updownarrow$ which are placed adjacent to the gauge groups in Table (4.3) indicate how the gauge group acts on the multiplet. Indeed the orientation of the multiplet has no strict significance and follows from our lingual habits. As we usually reserve the word up and down for the eigenvalues of isospin, so the multiplet is oriented vertically with respect to $SU(2)_{L,R}$. The $SU(4)$ symmetry acts then horizontally.

The indices $r, g, b, 4$ stand for color where the fourth color is denoted by 4. This assignment is both C and P invariant. The conjugated fermions transform under $\bar{4}$ of $SU(4)$. In Table (4.3) it is seen that we haven't put a bar on 1 in the lower part of the table. Actually it is unnecessary, since 1 denotes a singlet. In an $SU(n)$ representation , say n or $\bar{n}$, which are for fermions and their conjugates respectively, the representation matrices of $\bar{n}$ are obtainable from those of n. $-\lambda^*$ and λ are the representation matrices of $\bar{n}$ and n respectively. For $SU(2)$ the 2 and the $\bar{2}$ are indeed equivalent. Therefore one can write 2 instead of $\bar{2}$. The price we have to pay is to flip the fermions in the multiplet, because only the third component is different by an overall minus sign. To account for the usual spin-$1/2$ coupling it is then good to put a minus sign in front of any of the two fermions in the doublet. This is illustrated in Table (4.4). We have consulted to this trick while we want to use only the 2 or the usual pauli matrices.

$$U_i = \begin{bmatrix} \mathbf{U}_i & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \mathbf{U}_i & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\mathbf{U}_i^* & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\mathbf{U}_i^* & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \mathbf{U}_i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \mathbf{U}_i & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -\mathbf{U}_i^* & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\mathbf{U}_i^* \end{bmatrix} \equiv \begin{bmatrix} 4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \bar{4} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \bar{4} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \bar{4} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \bar{4} \end{bmatrix}$$

Tab. 4.2: Matrix representation of the $SU(4)$ group in $SO(10)$

The same fermion assignment can be vertically written as in eq. (4.10). The assignment automatically produces $B-L$ numbers and the correct electric charges. The decomposition with respect to the maximal subgroups should be considered in the first place [46]. For $SU(4)\times SU(2)_L\times SU(2)_R$, we obtain

$$\begin{aligned} 16_L &= (4,2,1)+(\bar{4},1,2) \\ 16_R &= (4,1,2)+(\bar{4},2,1) \end{aligned} \tag{4.13}$$

The 16 will have a decomposition with respect to $SU(3)\times SU(2)_L\times SU(2)_R\times U(1)_{B-L}$ as

$$\begin{aligned} 16_L &= (3,2,1)_{+1/3}+(1,2,1)_{-1}+(\bar{3},1,2)_{-1/3}+(\bar{1},1,2)_1 \\ 16_R &= (3,1,2)_{+1/3}+(1,1,2)_{-1}+(\bar{3},2,1)_{-1/3}+(\bar{1},2,1)_1 \end{aligned} \tag{4.14}$$

Disregarding $SU(2)_R$ the above decomposition can be done under $SU(3)\times SU(2)_L\times U(1)_R\times U(1)_{B-L}$ where $U(1)_R$ stands for the diagonal generator of $SU(2)_R$. For practical reasons it is not excluded.

$$\begin{aligned} 16_L &= (3,2)_{(0,\frac{1}{3})}+(1,2)_{(0,-1)}+(\bar{3},1)_{(+\frac{1}{2},-\frac{1}{3})}+(\bar{3},1)_{(-\frac{1}{2},-\frac{1}{3})}+(\bar{1},1)_{(\frac{1}{2},1)}+(\bar{1},1)_{(-\frac{1}{2},1)} \\ 16_R &= (3,1)_{(\frac{1}{2},\frac{1}{3})}+(3,1)_{(-\frac{1}{2},\frac{1}{3})}+(1,1)_{(\frac{1}{2},-1)}+(1,1)_{(-\frac{1}{2},-1)}+(\bar{3},2)_{(0,-\frac{1}{3})}+(\bar{1},2)_{(0,1)} \end{aligned} \tag{4.15}$$

Consequently the $SU(2)_R$ doublets appear as 2 singlets. The $U(1)_R$ and the $U(1)_{B-L}$ local gauge symmetries at this point can be swallowed by $U(1)_Y$ where obviously the rank is lowered by one. So that the fermions can be cast into the standard model assignment. Under $SU(3)\times SU(2)_L\times U(1)_Y$ both 16 decompose as

$$\begin{aligned} 16_L &= (3,2)_{1/6}+(1,2)_{-1/2}+(\bar{3},1)_{1/3}+(\bar{3},1)_{-2/3}+(\bar{1},1)_1+(\bar{1},1)_0 \\ 16_R &= (3,1)_{2/3}+(3,1)_{-1/3}+(1,1)_0+(1,1)_{-1}+(\bar{3},2)_{-1/6}+(\bar{1},2)_{1/2} \end{aligned} \tag{4.16}$$

where the $\frac{Y}{2}$ values are indicated explicitly as subscripts. The 16_L fermions are then decomposed into the following multiplets

$$\begin{pmatrix} d_i \\ -u_i \end{pmatrix}_L \quad \begin{pmatrix} e^- \\ -\nu_e \end{pmatrix}_L \quad (\ d_i^c\ )_L \quad (\ u_i^c\ )_L \quad (\ e^+\ )_L \quad (\ \nu^c\ )_L \tag{4.17}$$

The minus signs are remnant from Table (4.4). The 16_R fermions are decomposed as

$$(\ u_i\ )_R \quad (\ d_i\ )_R \quad (\ \nu\ )_R \quad (\ e^-\ )_R \quad \begin{pmatrix} d_i^c \\ -u_i^c \end{pmatrix}_R \quad \begin{pmatrix} e^+ \\ -\nu_e^c \end{pmatrix}_R \tag{4.18}$$

It is seen that 16_L is the CP transform of 16_R. In general the $SO(10)$ gauge interactions are both C and P invariant. Finally, if the electroweak symmetry breaks spontaneously down to $SU(3)_C\times U(1)_Q$ by means of a

	$\begin{pmatrix} u_r & u_g & u_b & \nu_4 \\ d_r & d_g & d_b & e_4 \end{pmatrix}_L$	$\begin{pmatrix} u_r & u_g & u_b & \nu_4 \\ d_r & d_g & d_b & e_4 \end{pmatrix}_R$
$SU(4)$	$4 \leftrightarrow$	$4 \leftrightarrow$
$SU(2)_L$	$2 \updownarrow$	$1 \updownarrow$
$SU(2)_R$	$1 \updownarrow$	$2 \updownarrow$
	$\begin{pmatrix} u^c_r & u^c_g & u^c_b & \nu^c_4 \\ d^c_r & d^c_g & d^c_b & e^c_4 \end{pmatrix}_R$	$\begin{pmatrix} u^c_r & u^c_g & u^c_b & \nu^c_4 \\ d^c_r & d^c_g & d^c_b & e^c_4 \end{pmatrix}_L$
$SU(4)$	$\bar{4} \leftrightarrow$	$\bar{4} \leftrightarrow$
$SU(2)_L$	$\bar{2} \updownarrow$	$1 \updownarrow$
$SU(2)_R$	$1 \updownarrow$	$\bar{2} \updownarrow$

Tab. 4.3: Transformation of fermions under $SU(4)_C \times SU(2)_L \times SU(2)_R$

Higgs doublet, as done in the electroweak theory, the fermion doublets in eq. (4.17) will transform as singlets under the relic symmetry. These singlets are to us interesting because the 16 can also be utilized as a Higgs representation. Assigning a Higgs field to the 16 is formally similar to what one does in the electroweak theory by assigning a Higgs field to the 2, but this is mainly considered in § 10. Under the $SU(3)_C \times U(1)_Q$ symmetry the electrically neutral singlets of the 16_L and 16_R are respectively

$$\left(\; -\nu_e \; \right)_L \;\; , \; \left(\; \nu^c \; \right)_L \tag{4.19}$$

and

$$\left(\; \nu \; \right)_R \;\; , \; \left(\; -\nu^c_e \; \right)_R \tag{4.20}$$

4.3.2 Charge Conjugation and Parity Transformation in $SO(10)$

The charge conjugation operator is denoted with C. We are looking for a suitable 32×32 dimensional matrix representation that fulfils the conditions :

$$\begin{aligned} C^{-1}\, U_i\, C &= -U_i^T \\ C^{-1}\, L_i\, C &= -L_i^T \\ C^{-1}\, R_i\, C &= -R_i^T \end{aligned} \tag{4.21}$$

The operation C can be easily guessed if we consider the U_i generators of $SU(4)$ in eq. (4.2) with $i = (1, \ldots, 15)$. Then under charge conjugation the 4's along the diagonal in eq. (4.2) should be transformed into into the $\bar{4}$'s and $vice\ versa$. So that the conjugate representation is obtained. The explicit form of C depends on the representation. Let us consider the Γ basis in § 2.3.3 and construct a product of the complex valued ones, so that we obtain

$$C = i\,\Gamma_1\,\Gamma_3\,\Gamma_5\,\Gamma_8\Gamma_{10} \tag{4.22}$$

	$\begin{pmatrix} d_r^c & d_g^c & d_b^c & e_4^c \\ -u_r^c & -u_g^c & -u_b^c & -\nu_4^c \end{pmatrix}_R$	$\begin{pmatrix} d_r^c & d_g^c & d_b^c & e_4^c \\ -u_r^c & -u_g^c & -u_b^c & -\nu_4^c \end{pmatrix}_L$
$SU(4)$	$\bar{4} \leftrightarrow$	$\bar{4} \leftrightarrow$
$SU(2)_L$	$2 \updownarrow$	$1 \updownarrow$
$SU(2)_R$	$1 \updownarrow$	$2 \updownarrow$

Tab. 4.4: The signs are flipped to get rid of the $\bar{2}$.

The explicit form of this particular charge conjugation matrix C is found as

$$C = \begin{pmatrix} 0 & A \\ -A & 0 \end{pmatrix}, \qquad A = \begin{pmatrix} 0 & 0 & 0 & -I \\ 0 & 0 & I & 0 \\ 0 & I & 0 & 0 \\ -I & 0 & 0 & 0 \end{pmatrix} \tag{4.23}$$

Here the I's are 4×4 unit matrices. Note that A operates similar to σ_2 which is used in case of 2 component Weyl spinors. But here we have a 16 component Weyl spinor. For both of the other basis given in § 2.3.1 and § 2.3.2 the charge conjugation matrix reads $C = i\,\Gamma_2\,\Gamma_4\,\Gamma_6\,\Gamma_8\Gamma_{10}$. Furthermore The Parity operator is denoted with P. We are looking for a suitable 32×32 dimensional matrix representation of P that fulfils the conditions :

$$\begin{aligned} P^{-1}\,L_i\,P &= R_i \\ P^{-1}\,R_i\,P &= L_i \\ P^{-1}\,U_i\,P &= U_i \end{aligned} \tag{4.24}$$

where $i = (1, 2, 3)$. Here P exchanges the left and right handed states. In other words, we look for a transformation that exchanges the generators of $SU(2)_L$ with those of $SU(2)_R$ and *vice versa*. The operation P can be guessed from the generators of the $SU(2)_L$ and $SU(2)_R$ groups given in eq. (4.1). Again the explicit form of P is representation dependent. If we consider the Γ basis in § 2.3.3 then a suitable choice is found as

$$P = i\Gamma_{10} = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \tag{4.25}$$

under the combined CP transformation, we have

$$\begin{aligned} (CP)^{-1}\;U_i\;(CP) &= -U_i^T\,, \\ (CP)^{-1}\;L_i\;(CP) &= -R_i^T\,, \\ (CP)^{-1}\;R_i\;(CP) &= -L_i^T\,, \end{aligned} \tag{4.26}$$

The P operator flips the chiral parts of the total spinor. We have

$$P\,\Psi = \Psi_P = P\begin{pmatrix} \Psi_L \\ \Psi_R \end{pmatrix} = \begin{pmatrix} \Psi_R \\ -\Psi_L \end{pmatrix} \tag{4.27}$$

As a result the P transformed spinor Ψ_P correctly transforms under the P transformed representation in eq. (4.24). Similarly we have

$$C\,\Psi = \Psi_C = C\begin{pmatrix} \Psi_L \\ \Psi_R \end{pmatrix} = \begin{pmatrix} A\Psi_R \\ -A\Psi_L \end{pmatrix} \tag{4.28}$$

so that Ψ_C transforms correctly under the C transformed representation in eq. (4.21). Under the combined CP transformation, we have

$$C\,P\,\Psi = \Psi_{CP} = \begin{pmatrix} A\,\Psi_L \\ -A\,\Psi_R \end{pmatrix} \tag{4.29}$$

Finally Ψ_{CP} correctly transforms under the CP transformed representation in eq. (4.26). Note that the CP transform of any fermion state in 16_L (or 16_R) is again in 16_L (or 16_R), whereas the C or P transform of a fermion in 16_L (or 16_R) is in 16_R (or 16_L). The $SO(10)$ interactions are CP and separately C and P invariant.

5. NEW INTERACTIONS IN $SO(10)$

5.1 The $(2,2,6)$ Fields

The complete spectrum of gauge bosons residing in the $SO(10)$ gauge group and their various properties have been elaborated in the previous parts. In this section, we will identify and sort out all the interactions within the $SO(10)$ theory. Since gauge interactions are meditated by gauge bosons, it is necessary to know the various vertices between fermions and gauge bosons. But once the symmetries are spontaneously broken, the gauge fields and thereby the currents undergo certain mixing. These physical currents will be studied later in § 13 and also in § 15.3. Here we study the currents prior to any SSB. We start our analysis by investigating the currents that couple to the $(2,2,6)$ fields. The relevant interaction Lagrangian reads

$$\begin{aligned}\mathcal{L}^{int}_{(226)} &= +ig\sqrt{2}\sum_{(226)} J_{\mu\ ab}W^{\mu}{}_{ab} = +i\frac{g}{\sqrt{2}}\sum_{(226)}\bar{\Psi}_L\gamma_\mu\Sigma_{ab}W_{ab}{}^{\mu}\Psi_L \\ &= +i\,g\sqrt{2}\left(J^A_\mu\cdot A^\mu + J^{A'}_\mu\cdot A'^\mu + J^Y_\mu\cdot Y^\mu + J^{Y'}_\mu\cdot Y'^\mu + h.c.\right) \\ &= +i\,g\sqrt{2}\,\bar{\Psi}_L\gamma_\mu\left(A^\mu\cdot D_A + A'^\mu\cdot D_{A'} + Y^\mu\cdot D_Y + Y'^\mu\cdot D_{Y'} + h.c.\right)\Psi_L\end{aligned} \tag{5.1}$$

Since the $(2,2,6)$ gauge fields lie in the coset, the above summation should be done over $a,b=(1,\ldots,10)$. This can be verified from the gauge fields presented in eq. (3.15). But in the above summation we select only the $(2,2,6)$ fields. From our previous expressions in eqs. (3.6) and (3.7), it becomes convenient to define the physical currents appearing in the second line in the above summation as

$$\begin{aligned}J^{A_\alpha} &= (\mathbf{J}_k - i\mathbf{J}_{k+6})/\sqrt{2} \qquad & J^{Y_\alpha} &= (\mathbf{J}_{k+12} - i\mathbf{J}_{k+18})/\sqrt{2} \\ J^{A'_\alpha} &= (\mathbf{J}_{k+3} - i\mathbf{J}_{k+9})/\sqrt{2} \qquad & J^{Y'_\alpha} &= (\mathbf{J}_{k+15} - i\mathbf{J}_{k+21})/\sqrt{2}\end{aligned} \tag{5.2}$$

where $k=(1,2,3)$. Here the currents J^{A_α},$J^{A'_\alpha}$,J^{Y_α} and $J^{Y'_\alpha}$ are color triplets with respect to $SU(3)$ with $\alpha=r,g,b$. Looking at the W_{ab} compositions of the $(2,2,6)$ gauge fields in eq. (3.15), we can find out the J_{ab} basis satisfying the above summation. We have

$$\begin{aligned}
\mathbf{J}_1 &= (J_{75}+J_{68})/\sqrt{2} & \mathbf{J}_{13} &= (J_{95}+J_{610})/\sqrt{2}\\
\mathbf{J}_2 &= (J_{37}+J_{48})/\sqrt{2} & \mathbf{J}_{14} &= (J_{39}+J_{410})/\sqrt{2}\\
\mathbf{J}_3 &= (J_{71}+J_{82})/\sqrt{2} & \mathbf{J}_{15} &= (J_{91}+J_{102})/\sqrt{2}\\
\mathbf{J}_4 &= (J_{59}+J_{610})/\sqrt{2} & \mathbf{J}_{16} &= (J_{75}+J_{86})/\sqrt{2}\\
\mathbf{J}_5 &= (J_{93}+J_{410})/\sqrt{2} & \mathbf{J}_{17} &= (J_{37}+J_{84})/\sqrt{2}\\
\mathbf{J}_6 &= (J_{19}+J_{102})/\sqrt{2} & \mathbf{J}_{18} &= (J_{71}+J_{28})/\sqrt{2}\\
\mathbf{J}_7 &= (J_{76}+J_{85})/\sqrt{2} & \mathbf{J}_{19} &= (J_{96}+J_{105})/\sqrt{2}\\
\mathbf{J}_8 &= (J_{74}+J_{38})/\sqrt{2} & \mathbf{J}_{20} &= (J_{94}+J_{310})/\sqrt{2}\\
\mathbf{J}_9 &= (J_{27}+J_{81})/\sqrt{2} & \mathbf{J}_{21} &= (J_{29}+J_{101})/\sqrt{2}\\
\mathbf{J}_{10} &= (J_{69}+J_{105})/\sqrt{2} & \mathbf{J}_{22} &= (J_{76}+J_{58})/\sqrt{2}\\
\mathbf{J}_{11} &= (J_{49}+J_{310})/\sqrt{2} & \mathbf{J}_{23} &= (J_{74}+J_{83})/\sqrt{2}\\
\mathbf{J}_{12} &= (J_{92}+J_{101})/\sqrt{2} & \mathbf{J}_{24} &= (J_{27}+J_{18})/\sqrt{2}
\end{aligned} \tag{5.3}$$

The third line in $\mathcal{L}^{int}_{(226)}$ above produces the desired currents

$$\begin{aligned}
J^{A_r}_\mu &= \frac{1}{\sqrt{2}}\left(+\bar{d}_{g\ L}\gamma_\mu d^c_{b\ L} - \bar{d}_{b\ L}\gamma_\mu d^c_{g\ L} + \bar{\nu^c}_L\gamma_\mu u_{rL} - \bar{u^c}_{rL}\gamma_\mu\nu_L\right)\\
J^{A_g}_\mu &= \frac{1}{\sqrt{2}}\left(-\bar{d}_{r\ L}\gamma_\mu d^c_{b\ L} + \bar{d}_{b\ L}\gamma_\mu d^c_{r\ L} + \bar{\nu^c}_L\gamma_\mu u_{gL} - \bar{u^c}_{gL}\gamma_\mu\nu_L\right)\\
J^{A_b}_\mu &= \frac{1}{\sqrt{2}}\left(+\bar{d}_{r\ L}\gamma_\mu d^c_{g\ L} - \bar{d}_{g\ L}\gamma_\mu d^c_{r\ L} + \bar{\nu^c}_L\gamma_\mu u_{bL} - \bar{u^c}_{bL}\gamma_\mu\nu_L\right)
\end{aligned} \tag{5.4}$$

$$
\begin{aligned}
J_\mu^{A'_r} &= \frac{1}{\sqrt{2}}\left(-\bar{u}_{g\ L}\gamma_\mu d^c_{b\ L} + \bar{u}_{b\ L}\gamma_\mu d^c_{g\ L} + \bar{\nu^c}_L\gamma_\mu d_{rL} - \bar{u^c}_{rL}\gamma_\mu e_L\right)\\
J_\mu^{A'_g} &= \frac{1}{\sqrt{2}}\left(+\bar{u}_{r\ L}\gamma_\mu d^c_{b\ L} - \bar{u}_{b\ L}\gamma_\mu d^c_{r\ L} + \bar{\nu^c}_L\gamma_\mu d_{gL} - \bar{u^c}_{gL}\gamma_\mu e_L\right)\\
J_\mu^{A'_b} &= \frac{1}{\sqrt{2}}\left(-\bar{u}_{r\ L}\gamma_\mu d^c_{g\ L} + \bar{u}_{g\ L}\gamma_\mu d^c_{r\ L} + \bar{\nu^c}_L\gamma_\mu d_{bL} - \bar{u^c}_{bL}\gamma_\mu e_L\right)
\end{aligned}
\tag{5.5}
$$

$$
\begin{aligned}
J_\mu^{Y_r} &= \frac{1}{\sqrt{2}}\left(-\bar{d}_{g\ L}\gamma_\mu u^c_{b\ L} + \bar{d}_{b\ L}\gamma_\mu u^c_{g\ L} + \bar{e^c}_L\gamma_\mu u_{rL} - \bar{d^c}_{rL}\gamma_\mu \nu_L\right)\\
J_\mu^{Y_g} &= \frac{1}{\sqrt{2}}\left(+\bar{d}_{r\ L}\gamma_\mu u^c_{b\ L} - \bar{d}_{b\ L}\gamma_\mu u^c_{r\ L} + \bar{e^c}_L\gamma_\mu u_{gL} - \bar{d^c}_{gL}\gamma_\mu \nu_L\right)\\
J_\mu^{Y_b} &= \frac{1}{\sqrt{2}}\left(-\bar{d}_{r\ L}\gamma_\mu u^c_{g\ L} + \bar{d}_{g\ L}\gamma_\mu u^c_{r\ L} + \bar{e^c}_L\gamma_\mu u_{bL} - \bar{d^c}_{bL}\gamma_\mu \nu_L\right)
\end{aligned}
\tag{5.6}
$$

$$
\begin{aligned}
J_\mu^{Y'_r} &= \frac{1}{\sqrt{2}}\left(+\bar{u}_{g\ L}\gamma_\mu u^c_{b\ L} - \bar{u}_{b\ L}\gamma_\mu u^c_{g\ L} - \bar{d^c}_{r\ L}\gamma_\mu e_L + \bar{e^c}_L\gamma_\mu d_{r\ L}\right)\\
J_\mu^{Y'_g} &= \frac{1}{\sqrt{2}}\left(-\bar{u}_{r\ L}\gamma_\mu u^c_{b\ L} + \bar{u}_{b\ L}\gamma_\mu u^c_{r\ L} - \bar{d^c}_{g\ L}\gamma_\mu e_L + \bar{e^c}_L\gamma_\mu d_{g\ L}\right)\\
J_\mu^{Y'_b} &= \frac{1}{\sqrt{2}}\left(+\bar{u}_{r\ L}\gamma_\mu u^c_{g\ L} - \bar{u}_{g\ L}\gamma_\mu u^c_{r\ L} - \bar{d^c}_{b\ L}\gamma_\mu e_L + \bar{e^c}_L\gamma_\mu d_{b\ L}\right)
\end{aligned}
\tag{5.7}
$$

All the above currents couple through the strength g. The four components in each current are shown as vertices in Fig. (5.1) where $\epsilon_{\alpha,\beta,\gamma} = -\epsilon_{\alpha,\gamma,\beta} = 1$ and the indices (α, β, γ) denote the colors (r, g, b) respectively. The same vertices are also shown in Fig. (5.2) where both quarks flow into the vertices or an antiquark and an antilepton flows into the vertex. These vertices can mediate various nucleon decays where the underlying mechanism is "quark + quark $\rightarrow$ antiquark + antilepton". This results in nucleon $\rightarrow$ meson + antilepton type processes which conserve $(B - L)$ at the vertex. The $(B - L)$ conserving processes that belong to these mechanism are given in Fig. (5.3). The conserved charges at the vertices in these diagrams follow from Table (3.1). Adding an appropriate third quark (fermion) line to each of the diagrams in Fig. (5.3) generates a nucleon decay process. Some examples are given in Fig. (5.4) where for each of the gauge bosons in the $(2, 2, 6)$ multiplet, one proton and one neutron decay process is generated. Note that at the vertices in Fig. (5.1) the Baryon number and the Lepton numbers are separately violated. Indeed the $(2, 2, 6)$ gauge fields carry the combination $B - L$ a way which is conserved [65][66].

5.2 *The* $(1, 1, 15)$ *Fields*

The $(1, 1, 15)$ gauge fields reside in the $SU(4)$ part of the $SO(10)$ group. We select out these fields and work out the currents coupling them. The interaction Lagrangian reads

$$
\begin{aligned}
\mathcal{L}^{int}_{(1,1,15)} &= +ig\sqrt{2}\sum_{(1,1,15)} J_{ab}\,W_{ab} = -i\frac{g}{\sqrt{2}}\sum_{(1,1,15)} \bar{\Psi}_L\gamma_\mu\Sigma_{ab}W_{ab}\Psi_L\\
&= +i\,g\sqrt{2}\left\{\left(J^X_{\mu\ \alpha}X^\mu{}_\alpha + h.c.\right) + J^\mu_{B-L}\frac{X_{\mu\,B-L}}{\sqrt{2}} + J_G\cdot G\right\}\\
&= +i\,g\sqrt{2}\,\bar{\Psi}_L\gamma_\mu\left\{G\cdot U_G + (X_\alpha\cdot U_{X_\alpha} + h.c.) + \frac{\sqrt{3}}{4}X_{B-L}\cdot U_{B-L}\right\}\Psi_L
\end{aligned}
\tag{5.8}
$$

Note that in the above expression we do not differentiate among the separate couplings and use a common g. The proper treatment of the couplings is postponed to § 5.5. The summation is done over $a, b = (1, \ldots, 6)$. These indices can be be read off from the eqs. (3.13) and (3.14) where the gluon fields G, the X_α fields and the X_{B-L} field are expressed in terms of the W_{ab} basis. For the physical currents in the second line in eq. (5.8) we introduce the following definitions

$$
\begin{aligned}
J^X_r &= J^{X\,\dagger}_r = (\mathcal{J}_9 - i\mathcal{J}_{10})/\sqrt{2}\\
J^X_g &= J^{X\,\dagger}_g = (\mathcal{J}_{11} - i\mathcal{J}_{12})/\sqrt{2}\\
J^X_b &= J^{X\,\dagger}_b = (\mathcal{J}_{13} - i\mathcal{J}_{14})/\sqrt{2}
\end{aligned}
\tag{5.9}
$$

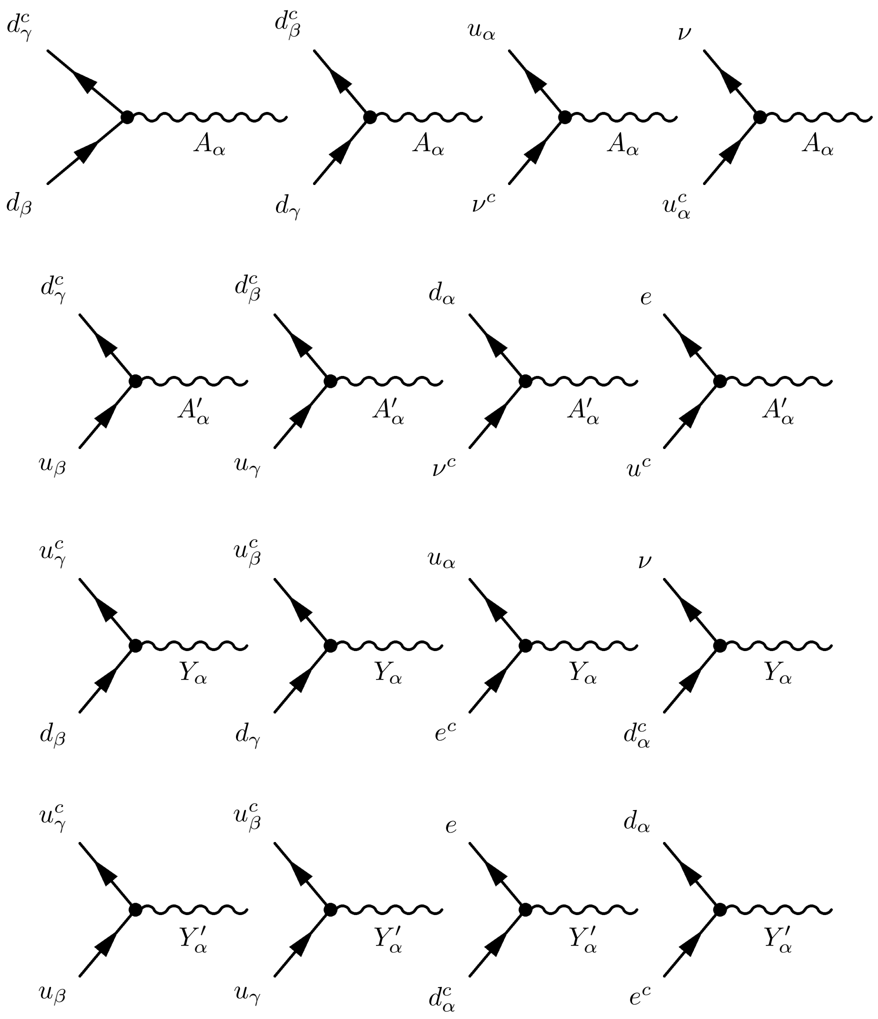

Fig. 5.1: A_α , A'_α , Y_α and Y'_α gauge bosons coupling to fermions.

Here $J_\alpha^{X\,\dagger}$ and J_α^X are charged raised and lowered currents respectively for $\alpha = (r, g, b)$ and form color triplets with respect to $SU(3)$. $\mathcal{J}_i$ for $i = (9, \ldots, 14)$ in terms of the basis J_{ab} reads

$$\begin{aligned}
\mathcal{J}_9 &= (J_{23} + J_{14})/\sqrt{2} \qquad & \mathcal{J}_{12} &= (J_{51} + J_{62})/\sqrt{2} \\
\mathcal{J}_{10} &= (J_{31} + J_{24})/\sqrt{2} \qquad & \mathcal{J}_{13} &= (J_{45} + J_{63})/\sqrt{2} \\
\mathcal{J}_{11} &= (J_{25} + J_{61})/\sqrt{2} \qquad & \mathcal{J}_{14} &= (J_{53} + J_{64})/\sqrt{2}
\end{aligned} \tag{5.10}$$

Single interaction terms in $\mathcal{L}_X^{int}$ can be explicitly found in that we act Ψ_L given in eq. (4.10) on the corresponding gauge term matrix of basis C. This yields

$$\mathcal{L}_X^{int} = +i\,\frac{g}{\sqrt{2}}\left\{-X_\alpha^\mu\left(\bar{d^c}_{\alpha\ L}\gamma_\mu e_L^c + \bar{u^c}_{\alpha\ L}\gamma_\mu \nu_L^c\right) + \bar{X}_\alpha^\mu\left(\bar{d}_{\alpha L}\gamma_\mu e_L + \bar{u}_{\alpha L}\gamma_\mu \nu_L\right) + h.c.\right\}$$

In these expressions we have e.g. $\bar{d} = d^\dagger\gamma^0$ and similarly $\bar{d^c} = (d^c)^\dagger\gamma^0$ by definition where γ_0 belongs to the dirac representation. The X_r, X_b and X_g bosons couple to the following currents respectively

$$\begin{aligned}
J^X_{r\ \mu} &= \frac{1}{\sqrt{2}}\left(-\bar{d^c}_{r\ L}\gamma_\mu e_L^c - \bar{u^c}_{r\ L}\gamma_\mu\nu_L^c + \bar{e}_L\gamma_\mu d_{rL} + \bar{\nu}_L\gamma_\mu u_{rL}\right) \\
J^X_{b\ \mu} &= \frac{1}{\sqrt{2}}\left(-\bar{d^c}_{b\ L}\gamma_\mu e_L^c - \bar{u^c}_{b\ L}\gamma_\mu\nu_L^c + \bar{e}_L\gamma_\mu d_{bL} + \bar{\nu}_L\gamma_\mu u_{bL}\right) \\
J^X_{g\ \mu} &= \frac{1}{\sqrt{2}}\left(-\bar{d^c}_{g\ L}\gamma_\mu e_L^c - \bar{u^c}_{g\ L}\gamma_\mu\nu_L^c + \bar{e}_L\gamma_\mu d_{gL} + \bar{\nu}_L\gamma_\mu u_{gL}\right)
\end{aligned} \tag{5.11}$$

All the above currents couple through the strength g. The single components in the above currents are shown in Fig. (5.5). The X_α bosons are often called lepto-quarks because they mediate the transition of a quark into a lepton. At the vertex of these transitions $B - L$ is conserved. The general form of these vertices is lepton + anti-quark $\rightarrow$

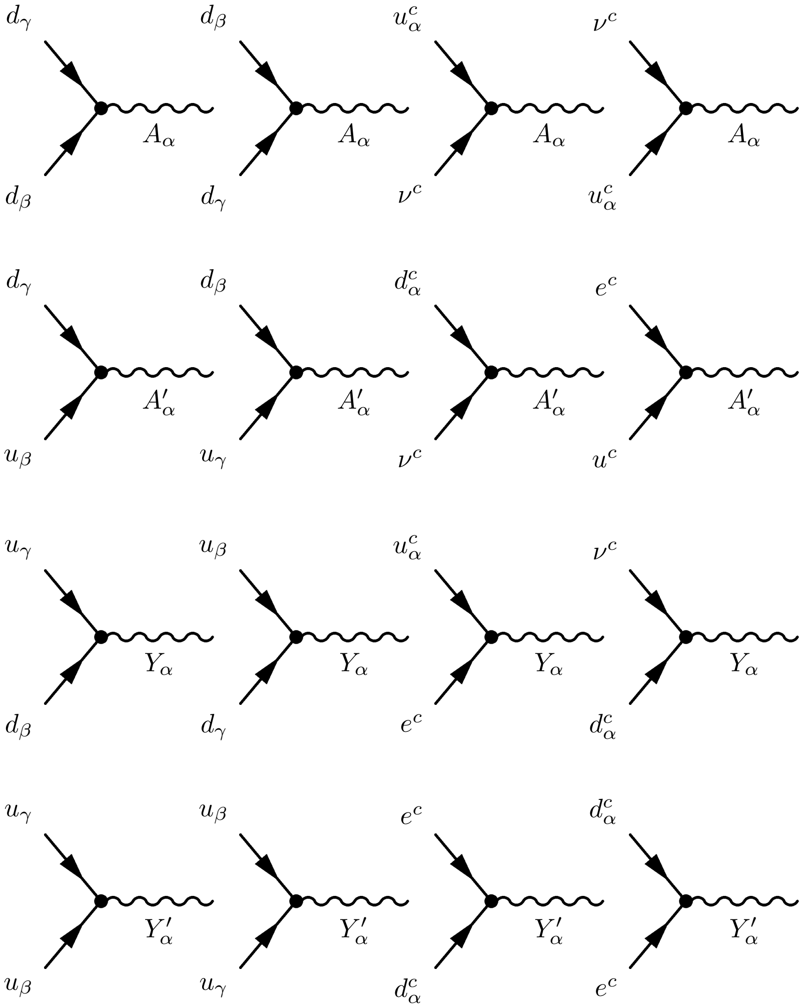

Fig. 5.2: A_α , A'_α , Y_α and Y'_α gauge bosons where lower fermions are lefthanded and upper are righthanded or vice versa

X. From the vertices in Fig. (5.5), it is seen that the X_α bosons can not mediate nucleon decays alone. We know that A_α has the same electric charge as X_α. Consequently through mixing the A_α and X_α fields one can generate nucleon decay processes. But these will be of $B-L$ violating character, because A_α and X_α have different $B-L$ numbers. Such processes are shown in Fig. (5.6) where the last diagram indicates a $B-L$ violating proton decay through $p \to \pi^+ + \nu$. The first two diagrams are neutron decays given through $n \to \pi^+ + e$ and $n \to \pi^0 + \nu$ respectively [67][68]. We proceed with the current coupling to the singlet field in the coset of $(1,1,15)$ which is the neutral X_{B-L} field. In conformity with our previous conventions the physical current will be defined as

$$J^\mu_{B-L} = \mathcal{J}_{15} = (J_{21} + J_{43} - J_{65})/(2\sqrt{6}) \tag{5.12}$$

If we act Ψ_L given in eq. (4.10) on the gauge term matrix of basis C, then the current coupling to X_{B-L} can be found. We have

$$\begin{aligned} J^{B-L}_\mu =& \frac{1}{2}\sqrt{\frac{3}{2}}\,(\,\frac{1}{3}\bar{u}_{\alpha L}\gamma_\mu u_{\alpha L} - \bar{\nu}_L\gamma_\mu\nu_L + \frac{1}{3}\bar{d}_{\alpha\ L}\gamma_\mu d_{\alpha\ L} - \bar{e}_L\gamma_\mu e_L \\ & - \frac{1}{3}\bar{u^c}_{\alpha L}\gamma_\mu u^c_{\alpha L} + \bar{\nu^c}_L\gamma_\mu\nu^c_L - \frac{1}{3}\bar{d^c}_{\alpha\ L}\gamma_\mu d^c_{\alpha\ L} + \bar{e^c}_L\gamma_\mu e^c_L\,) \end{aligned} \tag{5.13}$$

This current couples through the strength g. The remaining physical currents that we will elaborate couple to the fields in the $(1,1,8)$ part of the $(1,1,15)$ multiplet which are the gluon fields. The physical currents J_G in eq. (5.8) can conveniently be expressed through $\mathcal{J}_i$ for $i = (1,\ldots,8)$ as

$$\begin{array}{llll} G_{\bar{g}r} & = (G_{g\bar{r}})^\dagger = (W_{45} + W_{63} + i\,W_{53} + i\,W_{46})/2 & J_{r\bar{g}} & = (J_{\bar{r}g})^\dagger = (\mathcal{J}_1 - i\mathcal{J}_2)/\sqrt{2} \\ G_{\bar{b}r} & = (G_{b\bar{r}})^\dagger = (W_{52} + W_{61} + i\,W_{62} + i\,W_{15})/2 & J_{r\bar{b}} & = (J_{\bar{r}b})^\dagger = (\mathcal{J}_4 - i\mathcal{J}_5)/\sqrt{2} \\ G_{\bar{b}g} & = (G_{b\bar{g}})^\dagger = (W_{23} + W_{41} + i\,W_{31} + i\,W_{42})/2 & J_{g\bar{b}} & = (J_{\bar{g}b})^\dagger = (\mathcal{J}_6 - i\mathcal{J}_7)/\sqrt{2} \\ G_{\bar{r}r-\bar{b}b} & = (G_{\bar{r}r-\bar{b}b})^\dagger = (W_{21} + W_{43} + 2\,W_{65})/\sqrt{6} & J_{r\bar{r}-b\bar{b}} & = (J_{r\bar{r}-b\bar{b}})^\dagger = (\mathcal{J}_3\sqrt{3} + \mathcal{J}_8)/2 \\ G_{\bar{g}g-\bar{b}b} & = (G_{\bar{g}g-\bar{b}b})^\dagger = (W_{21} - 2\,W_{43} - W_{65})/\sqrt{6} & J_{g\bar{g}-b\bar{b}} & = (J_{g\bar{g}-b\bar{b}})^\dagger = (-\mathcal{J}_3\sqrt{3} + \mathcal{J}_8)/2 \end{array} \tag{5.14}$$

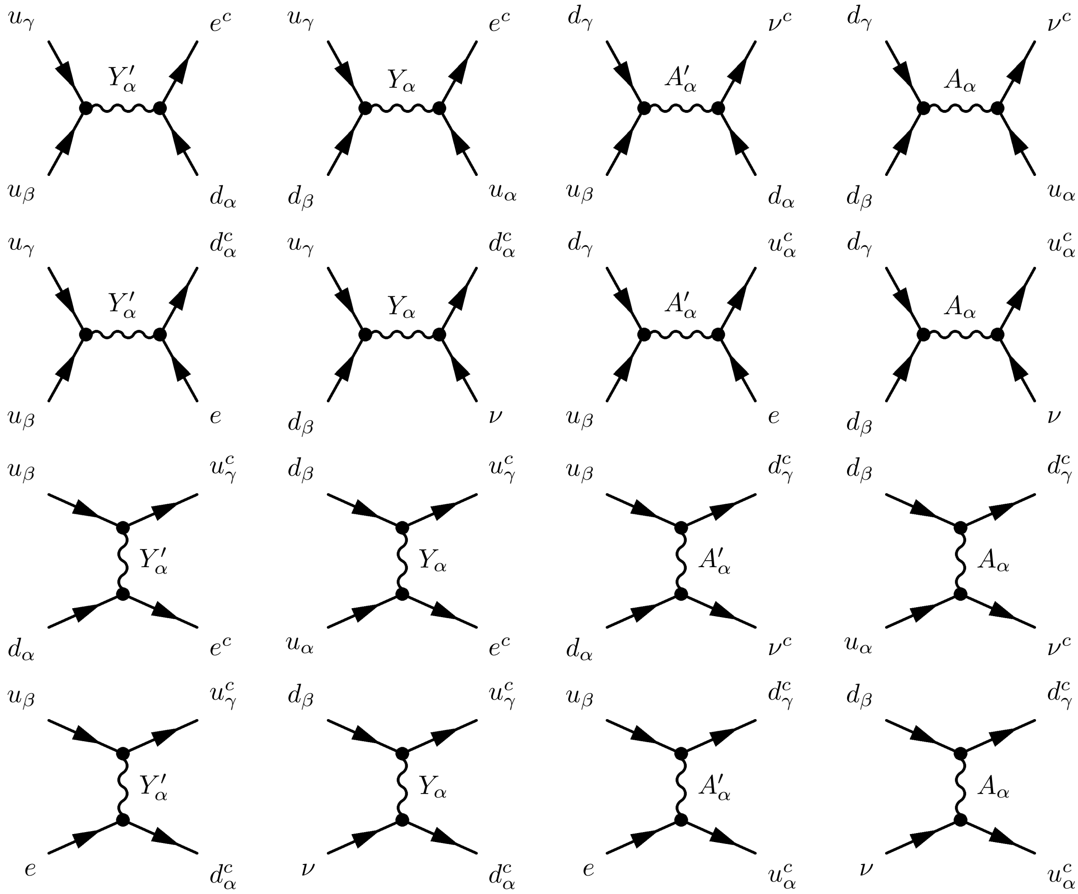

Fig. 5.3: Processes that give rise to nucleon decay through A_α , A'_α , Y_α and Y'_α gauge bosons.

In order to reflect the color nature of the gluon fields, we have relabelled the physical fields G and the physical currents J_G with the color states of the octet in Fig. (3.1) [69]. $\mathcal{J}_i$ for $i = (1, \ldots, 8)$ in terms of the basis J_{ab} reads

$$\begin{aligned} \mathcal{J}_1 &= (J_{45} + J_{36})/\sqrt{2} \qquad & \mathcal{J}_5 &= (J_{51} + J_{62})/\sqrt{2} \\ \mathcal{J}_2 &= (J_{53} + J_{46})/\sqrt{2} \qquad & \mathcal{J}_6 &= (J_{23} + J_{41})/\sqrt{2} \\ \mathcal{J}_3 &= (J_{65} + J_{43})/\sqrt{2} \qquad & \mathcal{J}_7 &= (J_{31} + J_{24})/\sqrt{2} \\ \mathcal{J}_4 &= (J_{52} + J_{61})/\sqrt{2} \qquad & \mathcal{J}_8 &= (2\,J_{21} + J_{34} + J_{65})/\sqrt{6} \end{aligned} \tag{5.15}$$

Note that the dot product in $J_G \cdot G$ sums over the internal octet space. The real fields V_i used in eq. (3.1) for say, $i = 1$ and $i = 2$ are then equal to $(G_{\bar{g}r} + G_{g\bar{r}})/\sqrt{2} = G_{\bar{g}r+g\bar{r}}$ and $-i(G_{\bar{g}r} - G_{g\bar{r}})/\sqrt{2} = -iG_{\bar{g}r-g\bar{r}}$ respectively. These are the elements of the $3 \otimes \bar{3} = 8 \oplus 1$ decomposition where 3 is the color triplet (r, g, b). Other color states of the octet can be found from above. However the color singlet 1 must be separately given as $(G_{\bar{r}r} + G_{g\bar{g}} + G_{b\bar{b}})/\sqrt{3}$ $= G_{\bar{r}r+g\bar{g}+b\bar{b}}$ which is not part of the $SU(3)$ interactions. This ninth element might be taken as the singlet Abelian part in $U(3) = SU(3) \otimes U(1)$ which promotes the color singlet to a massless boson mediating long ranged interactions. Such a singlet gluon is not very consistent with the current status of physics, and may not exist at all. This problem is not special to the $SO(10)$ theory as known. If we act Ψ_L given in eq. (4.10) on the gauge term matrix of basis C, then the currents coupling to G, which are very well known from QCD, are recovered. We have

$$\begin{aligned} J_{\bar{r}g} &= \frac{1}{\sqrt{2}}(\bar{u}_{r\,L}\gamma_\mu u_{g\,L} + \bar{d}_{r\,L}\gamma_\mu d_{g\,L} - \bar{u^c}_{g\,L}\gamma_\mu u^c_{r\,L} + \bar{d^c}_{g\,L}\gamma_\mu d^c_{r\,L}) \\ J_{\bar{r}b} &= \frac{1}{\sqrt{2}}(\bar{u}_{r\,L}\gamma_\mu u_{b\,L} + \bar{d}_{r\,L}\gamma_\mu d_{b\,L} - \bar{u^c}_{b\,L}\gamma_\mu u^c_{r\,L} + \bar{d^c}_{b\,L}\gamma_\mu d^c_{r\,L}) \\ J_{\bar{g}b} &= \frac{1}{\sqrt{2}}(\bar{u}_{g\,L}\gamma_\mu u_{b\,L} + \bar{d}_{g\,L}\gamma_\mu d_{b\,L} - \bar{u^c}_{b\,L}\gamma_\mu u^c_{g\,L} + \bar{d^c}_{b\,L}\gamma_\mu d^c_{g\,L}) \end{aligned} \tag{5.16}$$

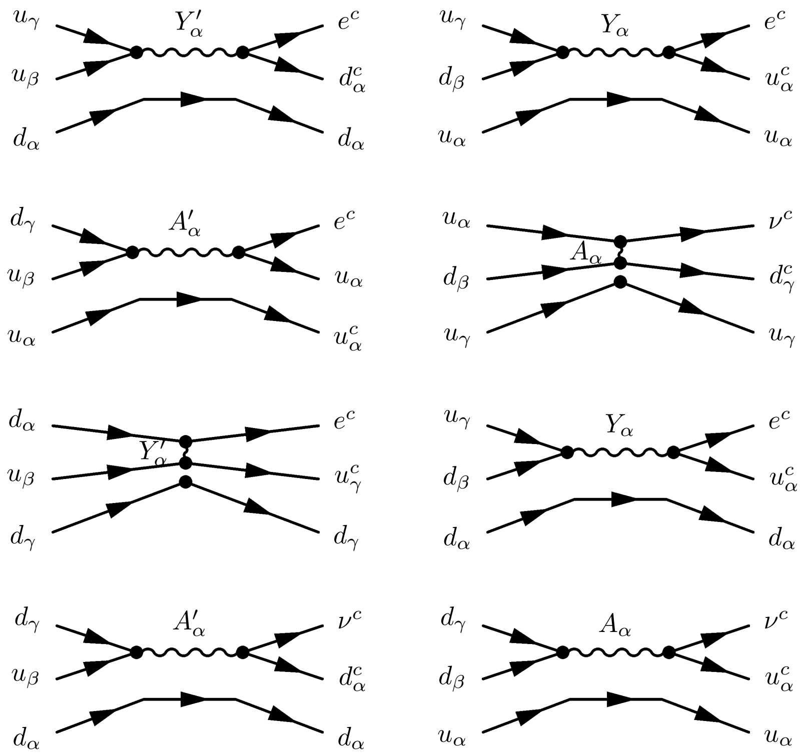

Fig. 5.4: Possible proton and neutron decay channels.

The color neutral currents take the form

$$
\begin{aligned}
J_{\bar r r - \bar b b} =& \frac{1}{\sqrt{3}}(\bar u_{r\,L}\gamma_\mu u_{r\,L} + \bar d_{r\,L}\gamma_\mu d_{r\,L} - \bar u_{b\,L}\gamma_\mu u_{b\,L} - \bar d_{b\,L}\gamma_\mu d_{b\,L} \\
& - \bar{u^c}_{r\,L}\gamma_\mu u^c_{r\,L} + \bar{d^c}_{r\,L}\gamma_\mu d^c_{r\,L} - \bar{u^c}_{b\,L}\gamma_\mu u^c_{b\,L} - \bar{d^c}_{b\,L}\gamma_\mu d^c_{b\,L}) \\
J_{\bar g g - \bar b b} =& \frac{1}{\sqrt{3}}(\bar u_{g\,L}\gamma_\mu u_{g\,L} + \bar d_{g\,L}\gamma_\mu d_{g\,L} - \bar u_{b\,L}\gamma_\mu u_{b\,L} - \bar d_{b\,L}\gamma_\mu d_{b\,L}) \\
& - \bar{u^c}_{g\,L}\gamma_\mu u^c_{g\,L} + \bar{d^c}_{g\,L}\gamma_\mu d^c_{g\,L} - \bar{u^c}_{b\,L}\gamma_\mu u^c_{b\,L} - \bar{d^c}_{b\,L}\gamma_\mu d^c_{b\,L})
\end{aligned}
\tag{5.17}
$$

All of the currents above couple through the strength g.

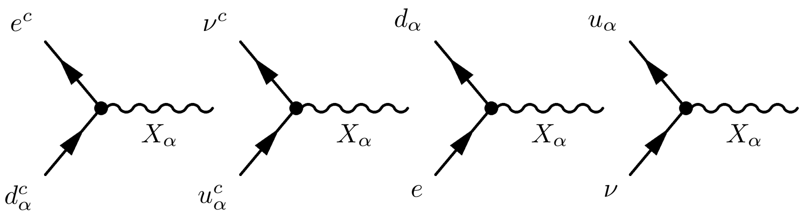

Fig. 5.5: X bosons

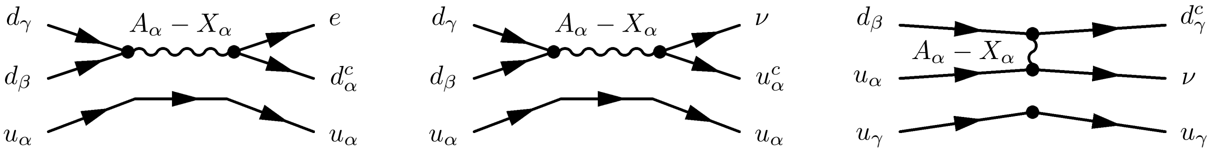

Fig. 5.6: $B - L$ violating processes

5.3 The $(1,3,1)$ and $(3,1,1)$ Fields

In the following we investigate the currents coupling to the gauge fields of the $SU(2)_L$ and $SU(2)_R$ gauge symmetries. The relevant interaction Lagrangian reads

$$
\begin{aligned}
\mathcal{L}^{int}_{(131)} + \mathcal{L}^{int}_{(311)} &= +i\,g\,\sqrt{2}\sum_{(a,b=7)}^{10} J_{ab}W_{ab} = +i\frac{g}{\sqrt{2}}\sum_{(a,b=7)}^{10} \bar{\Psi}_L\gamma_\mu\Sigma_{ab}W_{ab}\Psi_L \\
&= +i\,\sqrt{2}\,g\left(J^{\pm}_R W^{\pm}_R + J^0_R\frac{W^0_R}{\sqrt{2}} + J^{\pm}_L W^{\pm}_L + J^0_L\frac{W^0_L}{\sqrt{2}}\right) \\
&= +i\,\sqrt{2}\,g\,\bar{\Psi}_L\gamma_\mu\left(W^{\pm}_R R_{\pm} + \frac{W^0_R}{\sqrt{2}}R_0 + W^{\pm}_L L_{\pm} + \frac{W^0_L}{\sqrt{2}}L_0\right)\Psi_L
\end{aligned}
\tag{5.18}
$$

where we do not differentiate among the couplings of separate interactions and use a common g again. The coupling strengths g_L and g_R will be introduced later in § 5.5. The summation is done over the indices that belong to the $SO(4)$ multiplet. These indices can also be read off from the gauge fields presented in eq. (3.14). From the second line above it becomes convenient to define the physical currents as

$$
J^{\pm}_R = (J^1_R \mp iJ^2_R)/\sqrt{2}\,, \quad J^{\pm}_L = (J^1_L \pm iJ^2_L)/\sqrt{2}\,, \quad J^0_R = J^3_R\,, \quad J^0_L = J^3_L
\tag{5.19}
$$

Here the (J^1_R, J^2_R, J^3_R) and the (J^1_L, J^2_L, J^3_L) currents form $SU(2)_R$ and $SU(2)_L$ triplets respectively. Using our former expressions for the gauge fields in terms of W_{ab}, we can find the components of the physical currents in terms of the J_{ab}. We have

$$
\begin{aligned}
J^1_R &= (J_{79} + J_{810})/\sqrt{2}, \qquad J^1_L = (J_{79} + J_{108})/\sqrt{2} \\
J^2_R &= (J_{98} + J_{710})/\sqrt{2}, \qquad J^2_L = (J_{98} + J_{107})/\sqrt{2} \\
J^3_R &= (J_{87} + J_{910})/\sqrt{2}, \qquad J^3_L = (J_{87} + J_{109})/\sqrt{2}
\end{aligned}
\tag{5.20}
$$

The neutral and charged currents $J^{\pm,0}_R$ and $J^{\pm,0}_L$ respectively coupling to the $W^{\pm,0}_R$ and $W^{\pm,0}_L$ gauge fields given in eq. (5.18) are explicitly found as

$$
\begin{aligned}
J^+_L &= \frac{1}{\sqrt{2}}(\bar{d}_{\alpha\ L}\gamma_\mu u_{\alpha\ L} + \bar{e}_L\gamma_\mu\nu_L) \\
J^-_L &= \frac{1}{\sqrt{2}}(\bar{u}_{\alpha\ L}\gamma_\mu d_{\alpha\ L} + \bar{\nu}_L\gamma_\mu e_L) \\
J^-_R &= \frac{1}{\sqrt{2}}(-\bar{d^c}_{\alpha\ L}\gamma_\mu u^c_{\alpha\ L} - \bar{e^c}_L\gamma_\mu\nu^c_L) \\
J^+_R &= \frac{1}{\sqrt{2}}(-\bar{u^c}_{\alpha\ L}\gamma_\mu d^c_{\alpha\ L} - \bar{\nu^c}_L\gamma_\mu e^c_L) \\
J^0_R &= \frac{1}{2}(-\bar{u^c}_{\alpha\ L}\gamma_\mu u^c_{\alpha\ L} - \bar{\nu^c}_L\gamma_\mu\nu^c_L + \bar{d^c}_{\alpha\ L}\gamma_\mu d^c_{\alpha\ L} + \bar{e^c}_L\gamma_\mu e^c_L) \\
J^0_L &= \frac{1}{2}(\ \bar{u}_{\alpha\ L}\gamma_\mu u_{\alpha\ L}\ + \bar{\nu}_L\gamma_\mu\nu_L\ - \bar{d}_{\alpha\ L}\gamma_\mu d_{\alpha\ L}\ - \bar{e}_L\gamma_\mu e_L)
\end{aligned}
\tag{5.21}
$$

All the above currents couple with the strength g. The vertices described by the above currents are shown in Fig. (5.7). It is seen from the expressions of these isospin currents that the fermions f coupling to W_L are purely left-handed, and the fermions coupling to W_R are purely right-handed, where it is appropriate to replace $(f^c)_L$ with $(f_R)^c$ in the currents for $J^{\pm,0}_R$. As it will become clear later, certain Higgs scalars presented in § 6.2 and § 7.2 can spontaneously give *not only* masses to the $W^{\pm}_L$ and $W^{\pm}_R$ bosons, but can *also* lead them to mix. The resulting $W^{\pm}_L - W^{\pm}_R$ mixing is indeed a mass eigenstate. The details about the mixing depend on our specific choice of the Higgs sector which is presented in § 11.1 . The resulting mixing will be presented later in § 11.3.2.

5.4 The Electromagnetic Current

Let us consider the 3 neutral currents J^0_L, J^0_R and J_{B-L} which couple to the neutral gauge fields W^0_L, W^0_R and X_{B-L} respectively. These currents were given in eqs. (5.21) and (5.13). They add up to the electromagnetic current J_Q. We have

$$
\begin{aligned}
J_Q = J^0_L + J^0_R + \sqrt{\frac{2}{3}}J_{B-L} = (&-\frac{2}{3}\bar{u^c}_{\alpha\ L}\gamma_\mu u^c_{\alpha\ L} + \frac{1}{3}\bar{d^c}_{\alpha\ L}\gamma_\mu d^c_{\alpha\ L} + \bar{e^c}_L\gamma_\mu e^c_L \\
&+ \frac{2}{3}\bar{u}_{\alpha L}\gamma_\mu u_{\alpha L} - \frac{1}{3}\bar{d}_{\alpha\ L}\gamma_\mu d_{\alpha\ L} - \bar{e}_L\gamma_\mu e_L)
\end{aligned}
\tag{5.22}
$$

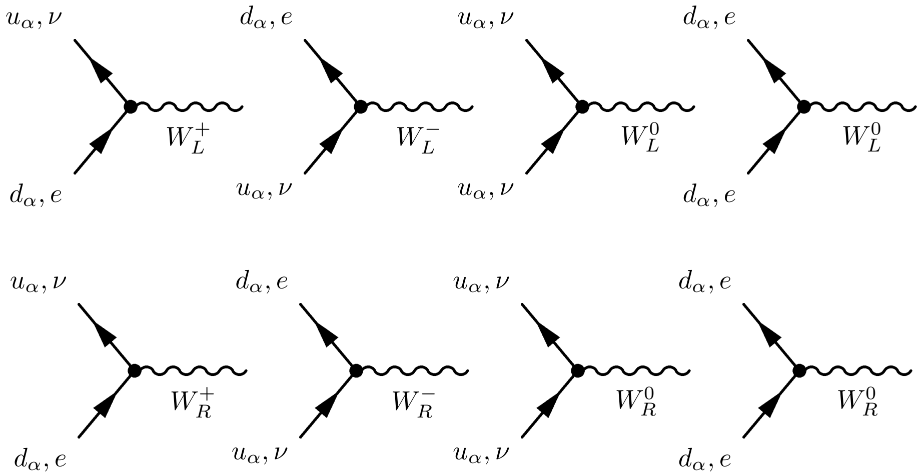

Fig. 5.7: $W_L^{\pm,0}$ and $W_R^{\pm,0}$ bosons coupling to fermions. In the upper half all fermions are left handed and in the lower half all fermions are right handed.

We know that the electromagnetic current J_Q couples to the photon. The physical photon however can only be a product of a spontaneous symmetry breakdown which correctly mixes the W_L^0, W_R^0 and X_{B-L} gauge fields into a massless eigenstate. Something similar happens in the electroweak theory. Let us write for the moment the following Lagrangian of electromagnetic interactions:

$$\mathcal{L}_Q^{int} = +i\,\sqrt{2}\,e\,\bar{\Psi}_L\gamma_\mu \left(\sqrt{\frac{8}{3}}\,\frac{A_Q}{\sqrt{2}}\cdot U_Q\right)\Psi_L \tag{5.23}$$

where A_Q is the electromagnetic gauge field and U_Q is the normalized electric charge generator stated as

$$U_Q = \sqrt{\frac{3}{8}}(L_3 + R_3 + \sqrt{\frac{2}{3}}U_{15}) = \sqrt{\frac{3}{8}}Q \tag{5.24}$$

Here Q is the eigenvalue operator for the electric charge. For any of the representations of Σ given in § 2 together with the corresponding Ψ_L, the above interaction Lagrangian $\mathcal{L}_Q^{int}$ delivers exactly the same electromagnetic current given in eq. (5.22) which was straightforwardly found through adding up the neutral currents. This is no surprise, because the U_Q generator defines an Abelian subgroup $U(1)_Q$, that can be embedded into the $SO(10)$ gauge group. In any spontaneous symmetry breakdown that preserves the $U(1)_Q$ symmetry, the A_Q field will appear as a massless mixing of the W_L^0, W_R^0 and X_{B-L} fields, which should be identified with the photon of the electroweak theory. This will be elaborated in great deal after we implement the spontaneous symmetry breakdown in the $SO(10)$ theory. The same also applies to the hypercharge gauge field and the hypercharge current.

5.5 Defining Separate Couplings Strengths in $SO(10)$

In this part we will restate the gauge term in eq. (4.1) in such away that it expresses all of the separate coupling strengths of interactions in the $SO(10)$ theory. This is important, because the mixing of separate gauge fields are given through expression that depend on the separate coupling strengths as well as the vacuum expectation values. In this respect, we move the single coupling strength g of $SO(10)$ into the gauge term. We have

$$\begin{aligned} +i\frac{g}{\sqrt{2}}W^{ab}\,\Sigma_{ab} = +\,i\,\sqrt{2}\,\Big\{ & g\,G\cdot U_G + \;g\;(X_\alpha\cdot U_{X_\alpha} + h.c.) + g\,\frac{X_{B-L}}{\sqrt{2}}\cdot U_{15} + g\,W_L^\pm L_\pm + g\,\frac{W_L^0}{\sqrt{2}}L_0 \\ & + g\,W_R^\pm R_\pm + g\,\frac{W_R^0}{\sqrt{2}}R_0 + g\,(D_{A_\alpha}\cdot A_\alpha + D_{A'_\alpha}\cdot A'_\alpha + D_{Y_\alpha}\cdot Y_\alpha + D_{Y'_\alpha}\cdot Y'_\alpha + h.c.)\Big\} \end{aligned} \tag{5.25}$$

The diagonal generators R_0 , L_0 and U_{15} and all other generators of $SO(10)$ were normalized to $Tr[(\ldots)^2] = 4$. In the above expression, we have to replace U_{15} with the corresponding eigenvalue operator $\frac{B-L}{2}$. Because the latter one is physical. From the other side the gauge term should remain unaffected by this replacement. Therefor a constant should be introduced to compensate the mismatch. The product of this constant with the coupling strength g will be defined as the respective coupling strength of the $U(1)_{B-L}$ interaction. Basically this procedure will be applied to all the other Abelian parts as well. Note that L_0, R_0 are already equal to their eigenvalue operators and are simultaneously normalized to 4. We introduce some coefficients and the new coupling strengths as follows

$$\begin{aligned}
Tr\left[(R_0\, C_R)^2\right] &= 4 \;\;\rightarrow\;\; C_R = 1\,, && g_R = g\; C_R \\
Tr\left[(L_0\, C_L)^2\right] &= 4 \;\;\rightarrow\;\; C_L = 1\,, && g_L = g\; C_L \\
Tr\left[\left(C_{B-L}\frac{U_{B-L}}{2}\right)^2\right] &= 4 \;\;\rightarrow\;\; C_{B-L} = \sqrt{\frac{3}{2}}\,, && g_{B-L} = g\; C_{B-L} \\
Tr\left[(Y\, C_Y)^2\right] &= 4 \;\;\rightarrow\;\; C_Y = \sqrt{\frac{3}{5}}\,, && g_Y = g\; C_Y \\
Tr\left[(Q\, C_Q)^2\right] &= 4 \;\;\rightarrow\;\; C_Q = \sqrt{\frac{3}{8}}\,, && e = g\; C_Q
\end{aligned} \tag{5.26}$$

Now we must replace the diagonal generators R_0 , L_0 and U_{15} with the equivalent expressions of $C_R\, R_0$, $C_L\, L_0$ and $C_{B-L}\frac{U_{B-L}}{2}$ respectively. Each time we absorb the coefficients C_L, C_R, C_{B-L} into the coupling strength g of $SO(10)$ and define the corresponding new coupling strength, so that g_R is the coupling strength for the $SU(2)_R$ interaction and g_L is the coupling strength of the $SU(2)_L$ weak isospin interaction. Also the g_{B-L} coupling is for the $U(1)_{B-L}$ interaction. Through these definitions the above gauge term can be restated as

$$\begin{aligned}
+i\frac{g}{\sqrt{2}}W^{ab}\,\Sigma_{ab} \equiv +\,i\,\sqrt{2}\,\Big\{ & g\, G\cdot U_G + \; g\;(X_\alpha\cdot U_{X_\alpha} + h.c.) + g_{B-L}\,\frac{X_{B-L}}{\sqrt{2}}\cdot\frac{U_{B-L}}{2} + g_L\, W_L^{\pm} L_{\pm} + g_L\,\frac{W_L^0}{\sqrt{2}} L_0 \\
& + g_R\, W_R^{\pm} R_{\pm} + g_R\,\frac{W_R^0}{\sqrt{2}} R_0 + g\,(D_{A_\alpha}\cdot A_\alpha + D_{A'_\alpha}\cdot A'_\alpha + D_{Y_\alpha}\cdot Y_\alpha + D_{Y'_\alpha}\cdot Y'_\alpha + h.c.)\Big\}
\end{aligned} \tag{5.27}$$

This is the gauge term that we will use as we implement the spontaneous symmetry breakdown in the $SO(10)$ theory. For further use, it is also necessary to know the C_Y and the C_Q coefficient related with the hyper charge and the electric charge respectively. These are given in eq. (5.26) where e is the coupling strength of the $U(1)_Q$ interaction and g_Y is the coupling strength of the $U(1)_Y$ interaction. The eigenvalue operator for hypercharge is defined as $Y = R_3 + \frac{U_{B-L}}{2}$. Some useful relations among the coupling strengths can be immediately derived. We have

$$\frac{1}{e^2} = \frac{1}{g_L^2} + \frac{1}{g_Y^2}\;\;, \qquad \frac{1}{g_Y^2} = \frac{1}{g_R^2} + \frac{1}{g_{B-L}^2} \quad\rightarrow\quad \frac{1}{e^2} = \frac{1}{g_L^2} + \frac{1}{g_R^2} + \frac{1}{g_{B-L}^2} \tag{5.28}$$

These relations are valid independently of how the generators in $SO(10)$ are normalized.

6. THE HIGGS MULTIPLETS: THE (10)-REPRESENTATION

6.1 The Structure

The 10 is the collection of the Γ_i basis with $i = (1, \ldots, 10)$ which also generates the $SO(10)$ generators Σ. The Γ_i basis satisfies the anti-commutation relation in eq.(2.11). These are 32×32 hermitian matrices normalized to $Tr(\Gamma_i^2) = 32$. Let us define 10 real scalar fields ϕ_i and construct a linear sum over the Γ_i basis. We have

$$\frac{\Gamma_i \phi_i}{\sqrt{32}} = \frac{1}{4}\left(\mathbf{\Gamma} \cdot \Phi + \mathbf{\Gamma}^\dagger \cdot \Phi^\dagger\right) = \frac{1}{4}\begin{pmatrix} \Omega_{11} & \Omega_{12} \\ \Omega_{21} & \Omega_{22} \end{pmatrix}, \qquad \frac{1}{32}\, Tr\left[(\Gamma_i \phi_i)^2\right] = \sum_{i=1}^{10} \phi_i^2 \tag{6.1}$$

where $(\,)^2 = (\,)(\,)^\dagger$. The Ω's are again 16×16 blocks. The explicit content of these blocks depend on the explicit matrix representation of the Γ_i's. The charge raised fields as well as the $\mathbf{\Gamma}$ basis is defined in advance as

$$\begin{aligned} \Phi_j &= (\phi_{2j-1} + i\,\phi_{2j})/\sqrt{2}, \qquad \mathbf{\Gamma}_j = (\Gamma_{2j-1} - i\,\Gamma_{2j})/2 \in (2,2,1)\ , \qquad j = 4, 5 \\ \Phi_j &= (\phi_{2j} + i\,\phi_{2j-1})/\sqrt{2}, \qquad \mathbf{\Gamma}_j = (\Gamma_{2j} - i\,\Gamma_{2j-1})/2 \in (1,1,6)\ , \qquad j = 1, 2, 3 \end{aligned} \tag{6.2}$$

The fields defined in the upper line i.e., for $j = (4, 5)$ above fall into the $(2, 2, 1)$ multiplet and the fields in the lower line i.e., for $j = (1, 2, 3)$ fall into the $(1, 1, 6)$ multiplet of the 10, if decomposed with respect to $SU(2)_L \times SU(R) \times SU(4)_c$ [53]. This splitting of the indices into $j = (4, 5)$ and $j = (1, 2, 3)$ is a direct consequence of the choice that we had previously adapted in eq. (2.28). By using our former definitions of the generators L_3 and R_3 which are indeed expressions of $\Sigma's$ and thereby also expressible through the Γ's, one can arrive at the following commutation relations.

$$\begin{aligned} [\mathbf{\Gamma}_5, L_3] &= +\frac{1}{2}\,\mathbf{\Gamma}_5 & [\mathbf{\Gamma}_5, R_3] &= -\frac{1}{2}\,\mathbf{\Gamma}_5 \\ [\mathbf{\Gamma}_4, L_3] &= +\frac{1}{2}\,\mathbf{\Gamma}_4 & [\mathbf{\Gamma}_4, R_3] &= +\frac{1}{2}\,\mathbf{\Gamma}_4 \\ \left[\mathbf{\Gamma}_5^\dagger, L_3\right] &= -\frac{1}{2}\,\mathbf{\Gamma}_5^\dagger & \left[\mathbf{\Gamma}_5^\dagger, R_3\right] &= +\frac{1}{2}\,\mathbf{\Gamma}_5^\dagger \\ \left[\mathbf{\Gamma}_4^\dagger, L_3\right] &= -\frac{1}{2}\,\mathbf{\Gamma}_4^\dagger & \left[\mathbf{\Gamma}_4^\dagger, R_3\right] &= -\frac{1}{2}\,\mathbf{\Gamma}_4^\dagger \end{aligned} \tag{6.3}$$

We can directly also use the explicit basis C for Γ_i where $(i = 1, \ldots, 10)$. Beneath the above bi-doublet structure, $\mathbf{\Gamma}_4$ and $\mathbf{\Gamma}_5$ commute with all the generators U_i of $SU(4)$ which is actually assured by the splitting of the indices. Therefore these 4 scalar fields above decompose as a bi-doublet and four-color singlet. i.e., $(2, 2, 1)$ under the $SU(2)_L \times SU(2)_R \times SU(4)_c$ group and are explicitly

$$\begin{aligned} (\phi_9 + i\,\phi_{10})/\sqrt{2} &= \Phi_5(+\frac{1}{2}, -\frac{1}{2}, 0, 0, 0) \qquad (\phi_9 - i\,\phi_{10})/\sqrt{2} = \bar{\Phi}_5(-\frac{1}{2}, +\frac{1}{2}, 0, 0, 0) \\ (\phi_7 + i\,\phi_8)/\sqrt{2} &= \Phi_4(+\frac{1}{2}, +\frac{1}{2}, 0, 0, 0) \qquad (\phi_7 - i\,\phi_8)/\sqrt{2} = \bar{\Phi}_4(-\frac{1}{2}, -\frac{1}{2}, 0, 0, 0) \end{aligned} \tag{6.4}$$

The values in parenthesis indicate always weights of $SU(2)_L \times SU(2)_R \times SU(4)_c$. These weights are given in the order of L_3, R_3, U_3 , U_8, U_{15} respectively. $\mathbf{\Gamma}_j$ for $j = (1, 2, 3)$ commute with L_3 and R_3, which is again assured by the splitting of the indices. Consequently the Higgs fields generated with $j = (1, 2, 3)$ above are left and right isospin singlets but they form in the $U_3 - U_8 - U_{15}$ space a $(1, 1, 6)$ sextet. We have

$$\begin{aligned} \left[\mathbf{\Gamma}_1^\dagger, U_3\right] &= 0\ \mathbf{\Gamma}_1^\dagger, & \left[\mathbf{\Gamma}_2^\dagger, U_3\right] &= +\tfrac{1}{2}\ \mathbf{\Gamma}_2^\dagger, & \left[\mathbf{\Gamma}_3^\dagger, U_3\right] &= +\tfrac{1}{2}\ \mathbf{\Gamma}_3^\dagger \\ \left[\mathbf{\Gamma}_1^\dagger, U_8\right] &= +\tfrac{1}{\sqrt{3}}\ \mathbf{\Gamma}_1^\dagger, & \left[\mathbf{\Gamma}_2^\dagger, U_8\right] &= -\tfrac{1}{2\sqrt{3}}\ \mathbf{\Gamma}_2^\dagger, & \left[\mathbf{\Gamma}_3^\dagger, U_8\right] &= +\tfrac{1}{2\sqrt{3}}\ \mathbf{\Gamma}_3^\dagger \\ \left[\mathbf{\Gamma}_1^\dagger, U_{15}\right] &= +\tfrac{1}{\sqrt{6}}\ \mathbf{\Gamma}_1^\dagger, & \left[\mathbf{\Gamma}_2^\dagger, U_{15}\right] &= +\tfrac{1}{\sqrt{6}}\ \mathbf{\Gamma}_2^\dagger, & \left[\mathbf{\Gamma}_3^\dagger, U_{15}\right] &= -\tfrac{1}{\sqrt{6}}\ \mathbf{\Gamma}_3^\dagger \end{aligned} \tag{6.5}$$

$$
\begin{array}{lllllllll}
[\,\mathbf{\Gamma}_1, U_3] & = & 0 & \mathbf{\Gamma}_1, & [\,\mathbf{\Gamma}_2, U_3] & = & -\frac{1}{2} & \mathbf{\Gamma}_2, & [\,\mathbf{\Gamma}_3, U_3] \; = \; -\frac{1}{2} \quad \mathbf{\Gamma}_3 \\
[\,\mathbf{\Gamma}_1, U_8] & = & -\frac{1}{\sqrt{3}} & \mathbf{\Gamma}_1, & [\,\mathbf{\Gamma}_2, U_8] & = & +\frac{1}{2\sqrt{3}} & \mathbf{\Gamma}_2, & [\,\mathbf{\Gamma}_3, U_8] \; = \; -\frac{1}{2\sqrt{3}} \quad \mathbf{\Gamma}_3 \\
[\,\mathbf{\Gamma}_1, U_{15}] & = & -\frac{1}{\sqrt{6}} & \mathbf{\Gamma}_1, & [\,\mathbf{\Gamma}_2, U_{15}] & = & -\frac{1}{\sqrt{6}} & \mathbf{\Gamma}_2, & [\,\mathbf{\Gamma}_3, U_{15}] \; = \; +\frac{1}{\sqrt{6}} \quad \mathbf{\Gamma}_3^\dagger
\end{array} \tag{6.6}
$$

The Higgs fields of the sextet carry either $1/3$ or $-1/3$ of (fractional) electric charge. They carry neither left nor right isospin and are explicitly defined as

$$
\begin{array}{ll}
(\phi_2 + i\phi_1)/\sqrt{2} = \Phi_1(0, 0, \;\; 0\;, -\frac{1}{\sqrt{3}}\;, -\frac{1}{\sqrt{6}}), & (\phi_2 - i\phi_1)/\sqrt{2} = \bar{\Phi}_1(0, 0, \;\; 0\;, +\frac{1}{\sqrt{3}}, \;\; +\frac{1}{\sqrt{6}}) \\
(\phi_4 + i\phi_3)/\sqrt{2} = \Phi_2(0, 0, -\frac{1}{2}, +\frac{1}{2\sqrt{3}}, -\frac{1}{\sqrt{6}}), & (\phi_4 - i\phi_3)/\sqrt{2} = \bar{\Phi}_2(0, 0, +\frac{1}{2}, -\frac{1}{2\sqrt{3}}, +\frac{1}{\sqrt{6}}) \\
(\phi_6 + i\phi_5)/\sqrt{2} = \Phi_3(0, 0, -\frac{1}{2}, -\frac{1}{2\sqrt{3}}, +\frac{1}{\sqrt{6}}), & (\phi_6 - i\phi_5)/\sqrt{2} = \bar{\Phi}_3(0, 0, +\frac{1}{2}, +\frac{1}{2\sqrt{3}}, -\frac{1}{\sqrt{6}})
\end{array} \tag{6.7}
$$

where the numbers in the parenthesis are weights as before. The electric charge of the Higgs fields can be obtained from $Q = L_3 + R_3 + \frac{1}{2}(B - L)$ where the $B - L$ number equals to $2\sqrt{2/3}$ times the U_{15} weight.i.e., the last entry in the parenthesis. Various charges of the Higgs fields residing in the 10 are summarized in Table (6.1) and the sextet is illustrated in Fig. (6.2). We highlight the content of Ω_{12} by using the basis C. We obtain

$$
\left[
\begin{array}{cccccccccccccccc}
\bar\Phi_5^{\downarrow\uparrow} & \cdot & \cdot & \cdot & \bar\Phi_4^{\downarrow\downarrow} & \cdot & \cdot & \cdot & \cdot & -\Phi_1 & -\Phi_2 & -\Phi_3 & \cdot & \cdot & \cdot & \cdot \\
\cdot & \bar\Phi_5^{\downarrow\uparrow} & \cdot & \cdot & \cdot & \bar\Phi_4^{\downarrow\downarrow} & \cdot & \cdot & \Phi_1 & \cdot & \bar\Phi_3 & -\bar\Phi_2 & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \bar\Phi_5^{\downarrow\uparrow} & \cdot & \cdot & \cdot & \bar\Phi_4^{\downarrow\downarrow} & \cdot & \Phi_2 & -\bar\Phi_3 & \cdot & \bar\Phi_1 & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \bar\Phi_5^{\downarrow\uparrow} & \cdot & \cdot & \cdot & \bar\Phi_4^{\downarrow\downarrow} & \Phi_3 & \bar\Phi_2 & -\bar\Phi_1 & \cdot & \cdot & \cdot & \cdot & \cdot \\
\Phi_4^{\uparrow\uparrow} & \cdot & \cdot & \cdot & -\Phi_5^{\uparrow\downarrow} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -\Phi_1 & -\Phi_2 & -\Phi_3 \\
\cdot & \Phi_4^{\uparrow\uparrow} & \cdot & \cdot & \cdot & -\Phi_5^{\uparrow\downarrow} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \Phi_1 & \cdot & \bar\Phi_3 & -\bar\Phi_2 \\
\cdot & \cdot & \Phi_4^{\uparrow\uparrow} & \cdot & \cdot & \cdot & -\Phi_5^{\uparrow\downarrow} & \cdot & \cdot & \cdot & \cdot & \cdot & \Phi_2 & -\bar\Phi_3 & \cdot & \bar\Phi_1 \\
\cdot & \cdot & \cdot & \Phi_4^{\uparrow\uparrow} & \cdot & \cdot & \cdot & -\Phi_5^{\uparrow\downarrow} & \cdot & \cdot & \cdot & \cdot & \Phi_3 & \bar\Phi_2 & -\bar\Phi_1 & \cdot \\
\cdot & \bar\Phi_1 & \bar\Phi_2 & \bar\Phi_3 & \cdot & \cdot & \cdot & \cdot & -\Phi_5^{\uparrow\downarrow} & \cdot & \cdot & \cdot & -\bar\Phi_4^{\downarrow\downarrow} & \cdot & \cdot & \cdot \\
-\bar\Phi_1 & \cdot & -\Phi_3 & \Phi_2 & \cdot & \cdot & \cdot & \cdot & \cdot & -\Phi_5^{\uparrow\downarrow} & \cdot & \cdot & \cdot & -\bar\Phi_4^{\downarrow\downarrow} & \cdot & \cdot \\
-\bar\Phi_2 & \Phi_3 & \cdot & -\Phi_1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -\Phi_5^{\uparrow\downarrow} & \cdot & \cdot & \cdot & -\bar\Phi_4^{\downarrow\downarrow} & \cdot \\
-\bar\Phi_3 & -\Phi_2 & \Phi_1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -\Phi_5^{\uparrow\downarrow} & \cdot & \cdot & \cdot & -\bar\Phi_4^{\downarrow\downarrow} \\
\cdot & \cdot & \cdot & \cdot & \cdot & \bar\Phi_1 & \bar\Phi_2 & \bar\Phi_3 & -\Phi_4^{\uparrow\uparrow} & \cdot & \cdot & \cdot & \bar\Phi_5^{\downarrow\uparrow} & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & -\bar\Phi_1 & \cdot & -\Phi_3 & \Phi_2 & \cdot & -\Phi_4^{\uparrow\uparrow} & \cdot & \cdot & \cdot & \bar\Phi_5^{\downarrow\uparrow} & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & -\bar\Phi_2 & \Phi_3 & \cdot & -\Phi_1 & \cdot & \cdot & -\Phi_4^{\uparrow\uparrow} & \cdot & \cdot & \cdot & \bar\Phi_5^{\downarrow\uparrow} & \cdot \\
\cdot & \cdot & \cdot & \cdot & -\bar\Phi_3 & -\Phi_2 & \Phi_1 & \cdot & \cdot & \cdot & \cdot & -\Phi_4^{\uparrow\uparrow} & \cdot & \cdot & \cdot & \bar\Phi_5^{\downarrow\uparrow}
\end{array}
\right]
$$

where the dots simply denote zeros. $\Omega_{11} \equiv \Omega_{22}$ and are null blocks. But Ω_{21} is the Hermitian conjugate of Ω_{12}. In the above Higgs matrix the superscripts $\uparrow\uparrow$, $\uparrow\downarrow$, $\downarrow\uparrow$ and $\downarrow\downarrow$ are also highlighted and denote the left and right isospin states respectively.

In a spontaneous symmetry breaking which breaks $SU(2)_L \times SU(2)_R \times SU(4)$ down to $SU(2)_L \times SU(2)_R \times SU(3)_c \times U(1)_{B-L}$, the sextet of the 10 would decompose into a $3 + \bar{3}$. i.e., a $(1, 1, 3)$ and a $(1, 1, \bar{3})$ multiplet. These triplets are differentiated by their U_{15} weights, namely through the $\pm 1/\sqrt{6}$ value. They carry not only electric charge but also color. The gluons of the $SU(3)$ color group should remain massless and in addition the electric charge conservation should not be violated. Therefore none of the Φ fields in the $(1, 1, 6)$ sextet should ever be allowed to receive any vacuum expectation value (vev) in our calculations. As a result the sextet is physically uninteresting.

The physically interesting and useful part of the 10 is however the $(2, 2, 1)$ multiplet. Despite of the fact that the Φ_4 field and its conjugate are $SU(4)$ and thereby also $SU(3)$ singlets, they carry non-zero electric charge as was shown in Table 6.1. Therefore Φ_4 and its conjugate are no suitable Higgs fields because they break the $U(1)_Q$ symmetry. From their charges, it is seen that only Φ_5 and its conjugate may be allowed to receive some vev, because they do not violate the conservation of electric charge and $SU(3)$ color, or even the $SU(4)$ color symmetry. The gauge fields lying in the coset of $SU(4)/SU(3)$ will receive from the vev of Φ_5 no mass. Certain features of this vev are summarized in § 6.2. By using the Higgs fields of the 10 which were constructed from the product $\Phi_j \mathbf{\Gamma_j}$, we choose in the light of our analysis a suitable Higgs term to be employed in the Higgs mechanism. We have

$$\Phi_{221} \equiv \frac{1}{\sqrt{32}}\left(\Phi_5 \mathbf{\Gamma_5} + \Phi_5^\dagger \mathbf{\Gamma_5}^\dagger\right); \qquad Tr\left[(\Phi_{221})^2\right] = \Phi_5 \Phi_5^\dagger \tag{6.8}$$

here both Higgs fields belong to the $(2,2,1)$. From now on, we will always refer to the above two Higgs fields with the $(2,2,1)$ or equivalently with Φ_{221}. This term does break the following initial symmetries to the stated final symmetries

(i) $SU(4) \times SU(2)_L \times SU(2)_R$ down to $SU(4) \times U(1)_{L+R}$. To be explicit, the $U(1)_{L+R}$ is an Abelian group whose single generator is the $L_3 + R_3$. The $U(1)_{L+R}$ symmetry, operating on the 16_L (or 16_R) spinor of fermions is understood to yield the sum of the left and right isospin numbers of the corresponding fermions.

(ii) $SU(3) \times SU(2)_L \times SU(2)_R \times U(1)_{B-L}$ symmetry down to $SU(3) \times U(1)_{L+R} \times U(1)_{B-L}$ where $U(1)_{B-L}$ is the gauge group for $B-L$ interactions with generator proportional to U_{15}. Exactly speaking the $U_{B-L}/2$.

(iii) $SU(3) \times SU(2)_L \times U(1)_Y$ down to $SU(3) \times U(1)_Q$ where $U(1)_Q$ is the electromagnetic gauge group with the generator $Q = L_3 + R_3 + (B-L)/2$ and $U(1)_Y$ is the hypercharge gauge group with the eigenvalue operator $Y = R_3 + (B-L)/2$.

(iv) $SU(3) \times SU(2)_R \times U(1)_{Y'}$ down to $SU(3) \times U(1)_Q$ where the $U(1)_{Y'}$ is an Abelian group whose eigenvalue operator is $L_3 + (B-L)/2$. The $U(1)_{Y'}$ symmetry operating on the 16_L (or 16_R) spinor of fermions is understood to yield the sum of the L_3 and $(B-L)/2$ eigenvalues.

In all the above symmetry breaking, electric charge conservation is respected. We have

$$\begin{aligned}
&\mathbf{\Gamma}_5\ (+\frac{1}{2}, -\frac{1}{2}, 0, 0, 0) \to \Delta_L + \Delta_R = 0 = \Delta_Q, \quad \Delta_{B-L} \equiv 0 \ \ \to \Delta_Y = \Delta_R \\
&\mathbf{\Gamma}_5^\dagger\ (-\frac{1}{2}, +\frac{1}{2}, 0, 0, 0) \to \Delta_L + \Delta_R = 0 = \Delta_Q, \quad \Delta_{B-L} \equiv 0 \ \ \to \Delta_Y = \Delta_R
\end{aligned} \tag{6.9}$$

where in each line the Δ's indicate the amount of non-commutation of $\mathbf{\Gamma}_5$ (or its conjugate) with the diagonal generator of the respective symmetry. i.e., L_3, R_3, Y or $(B-L)$. In terms of the Γ_i matrices of basis C, the $\mathbf{\Gamma}_5^\dagger$ and $\mathbf{\Gamma}_5^\dagger$ assume the form

$$\mathbf{\Gamma}_5^\dagger = \begin{bmatrix} 0 & B \\ A & 0 \end{bmatrix}, \quad \mathbf{\Gamma}_5 = \begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix}, \quad A = \begin{bmatrix} \mathbb{1} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathbb{1} \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & \mathbb{1} & 0 & 0 \\ 0 & 0 & \mathbb{1} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{6.10}$$

here $\mathbb{1}$ is a 4×4 unit matrix.

6.2 Features of the Φ_{221}

Two other features that are worth discussing is how the Φ_{221} Higgs fields given in eq. (6.8) behave in the Yukawa sector and how they behave in the Higgs mechanism.

(i) Let us for brevity consider the most general case in which we break the $SO(10)$ gauge group only with the Higgs field $\Phi_5 \mathbf{\Gamma_5} + \Phi_5^\dagger \mathbf{\Gamma_5}^\dagger$. Then the resulting vev endows the following gauge bosons of the 45 with mass : W_L^+, W_L^0, W_L^-, W_R^+, W_R^0,W_R^-, A'_r, A'_g, A'_b and Y_r,Y_g,Y_b. *Remark* : It should be noted that if the above listed gauge bosons already have mass due to some other Higgs field they will gain additional masses up on the Φ_{221} breaking.

(ii) One can show that $\Phi_5 \mathbf{\Gamma_5} + \Phi_5^\dagger \mathbf{\Gamma_5}^\dagger$ induces mixing among certain gauge bosons. The gauge bosons in (i) mix: $W_L^- - W_R^-$, $W_L^+ - W_R^+$, $W_L^0 - W_R^0$, $A'_\alpha - Y_\alpha$, where the sign "$-$" illustrates mixing. A pure Φ_{221} breaking produces one massive and one massless mode for all the above mixed states. i.e., There will be a total number of six massive and six massless mixed gauge bosons. However the detailed nature of the mixing will be elaborated after the Higgs sector is fully appreciated.

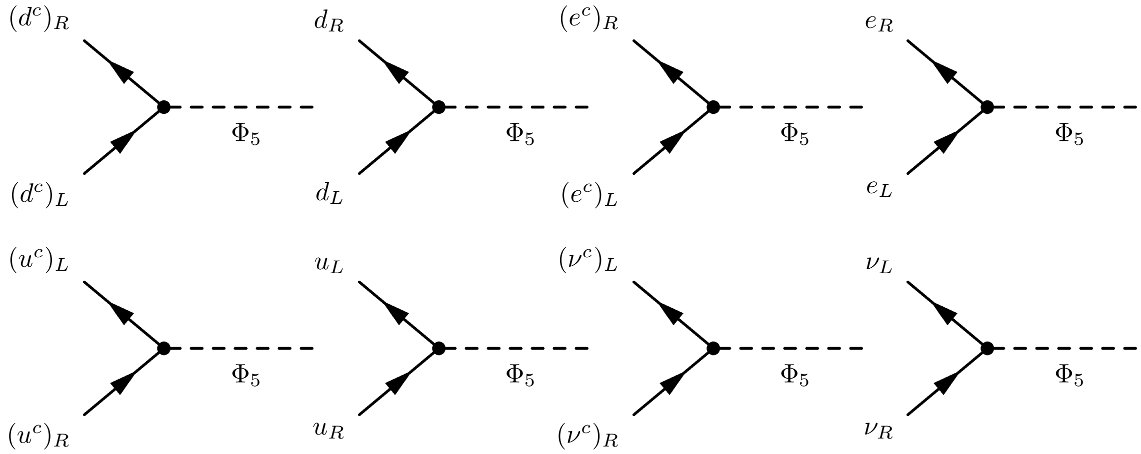

Fig. 6.1: The Scalar Φ_5 coupling to fermions, where Φ_5 is flowing into the vertex

(iii) Let us consider the Yukawa term $Y_{ij}\,\left(\bar{\Psi}_i\ \Phi_{221}\ \Psi_j\right)$, where $i,j=(1,2,3)$ are family indices, Y_{ij} are the Yukawa couplings and Ψ_i are the family spinors. The vacuum expectation values of the $\Phi_5\mathbf{\Gamma_5}+\Phi_5^\dagger\mathbf{\Gamma_5}^\dagger$ Higgs fields endow all the fermions with Dirac masses. The vacuum expectation values of the $\Phi_5\mathbf{\Gamma_5}+\Phi_5^\dagger\mathbf{\Gamma_5}^\dagger$ Higgs fields endow all the fermions with Dirac masses. The vertices at which the Higgs scalars couple to the fermions are shown in Fig. (6.1). The charges conserved at each vertex can be checked from Tables (4.3) and (4.4), (7.1). They produce equal masses for fermions.i.e., up-leptons and up-quarks get equal masses like $m_u=m_\nu$. Also down-leptons and down-quarks get equal masses like $m_d=m_e$. The equality of lepton and quark masses can be easily deduced from the uniformity of the matrix Ω which was shown in eq. (6.10). Finally the up fermions and down fermions get all the same mass like $m_u=m_d$, because Φ_5 and $\Phi_5^\dagger$ are conjugated to each other., i.e., they assume vev's that can differ at most by a phase.

Details concerning the above features will be considered later on again.

6.3 Weight Diagrams for the 10

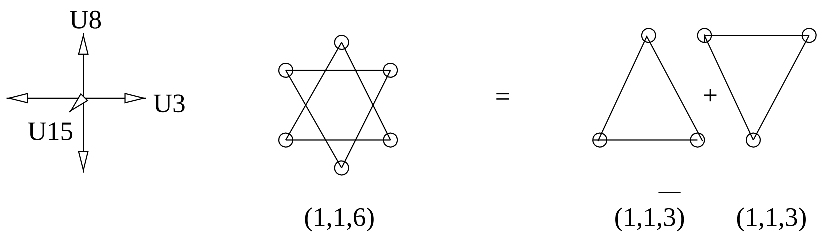

Fig. 6.2: The decomposition of the sextet in the 10 with respect to $SU(3)\times SU(2)_L\times SU(2)_R\times U(1)_{B-L}$. The L,R isospin weights are suppressed and U_{15} points out of page.

Charges of the 10 Higgs Fields											
(2,2,1)	Q	B-L	I_{3L}	I_{3R}	Y	(1,1,6)	Q	B-L	I_{3L}	I_{3R}	Y
$\bar{\Phi}_4^{\downarrow\downarrow}$	-1	0	-1/2	-1/2	-1	Φ_1	-1/3	-2/3	0	0	-2/3
$\bar{\Phi}_5^{\downarrow\uparrow}$	0	0	-1/2	+1/2	+1	Φ_2	-1/3	-2/3	0	0	-2/3
$\Phi_5^{\uparrow\downarrow}$	0	0	+1/2	-1/2	-1	Φ_3	+1/3	+2/3	0	0	+2/3
$\Phi_4^{\uparrow\uparrow}$	+1	0	+1/2	+1/2	+1	$\bar{\Phi}_1$	+1/3	+2/3	0	0	+2/3
						$\bar{\Phi}_2$	+1/3	+2/3	0	0	+2/3
						$\bar{\Phi}_3$	-1/3	-2/3	0	0	-2/3

Tab. 6.1: Charges of the scalar bosons in the 10 Higgs representation of $SO(10)$

7. THE HIGGS MULTIPLETS: THE (126)-REPRESENTATION

In this part we elaborate the 126 Higgs representation. Not all of these Higgs fields will be useful, only those which can be implemented in the spontaneous breakdown of $SO(10)$ over various intermediate symmetries down to $SU(3) \times U(1)_Q$, will be to us of physical importance. The 126 is spanned by the 5-products of the Γ_i basis of $SO(10)$ which were introduced in $\S$ 2.3. These 5-products include all the possible $\Gamma_i\Gamma_j\Gamma_k\Gamma_l\Gamma_m$ terms in which $i \neq j \neq k \neq l \neq m$. There will be 252 such products. But half of them will be related over Γ_{five} to the rest. This means that, there will be 126 Higgs fields and 126 charge conjugated Higgs fields, namely the $\overline{126}$. The expansion reads

$$\frac{1}{\sqrt{32}}\,\phi_{ijklm}\Gamma_i\Gamma_j\Gamma_k\Gamma_l\Gamma_m = \frac{1}{16}\left(\mathbf{\Gamma}\cdot\Phi + \mathbf{\Gamma}^\dagger\cdot\Phi^\dagger\right) = \left(\begin{array}{cc} \Omega_{11} & \Omega_{12} \\ \Omega_{21} & \Omega_{22} \end{array}\right) \tag{7.1}$$

where $\mathbf{\Gamma}$ are 126 linearly independent combination of the 5-products and Φ are the charge raised 126 fields. It is formally not pleasing to treat these fields over a single index like Φ_α with $\alpha = (1, \ldots, 126)$. Therefore we must develop a relatively practical way of labelling. We use two lower indices to denote left and right isospin states and an upper index to distinguish among those fields which posses the same left-right isospin state. The convention will be explained in some more detail later on. The complete list of the Higgs fields Φ and the $\mathbf{\Gamma}$'s of the 126 expanded in terms of the real valued scalar fields ϕ_{ijklm} and the 5-products $\Gamma_i\Gamma_j\Gamma_k\Gamma_l\Gamma_m$ respectively are given in Appendix B. As done before, the total 32×32 Higgs matrix will be truncated into 16×16 blocks of matrices as shown in eq. (7.1). To be illustrative, let us consider the Γ_i matrices of basis C to depict the matrix of Ω's. Then Ω_{11} and Ω_{22} turn out to be occupied fully by zeros. The Ω_{12} part is

$$\left[\begin{array}{cccccccccccccccc}
\lambda_1 & \Phi^{4}_{\downarrow\uparrow} & \Phi^{5}_{\downarrow\uparrow} & \Phi^{12}_{\downarrow\uparrow} & \lambda_7 & \Phi^{4}_{\downarrow\downarrow} & \Phi^{5}_{\downarrow\downarrow} & \Phi^{12}_{\downarrow\downarrow} & \Phi^{7}_{\hat{0}0} & \alpha_1 & \alpha_2 & \alpha_3 & \Phi^{7}_{\downarrow 0} & \Phi^{1}_{\downarrow 0} & \Phi^{2}_{\downarrow 0} & \Phi^{3}_{\downarrow 0} \\
\Phi^{1}_{\downarrow\uparrow} & \lambda_2 & \Phi^{6}_{\downarrow\uparrow} & \Phi^{13}_{\downarrow\uparrow} & \Phi^{1}_{\downarrow\uparrow} & \lambda_8 & \Phi^{6}_{\downarrow\downarrow} & \Phi^{13}_{\downarrow\downarrow} & \alpha_4 & \Phi^{8}_{\hat{0}0} & \alpha_{12} & \alpha_{11} & \Phi^{1}_{\downarrow 0} & \Phi^{8}_{\downarrow 0} & \Phi^{6}_{\downarrow 0} & \Phi^{5}_{\downarrow 0} \\
\Phi^{2}_{\downarrow\uparrow} & \Phi^{3}_{\downarrow\uparrow} & \lambda_3 & \Phi^{14}_{\downarrow\uparrow} & \Phi^{2}_{\downarrow\uparrow} & \Phi^{3}_{\downarrow\downarrow} & \lambda_9 & \Phi^{14}_{\downarrow\downarrow} & \alpha_5 & \alpha_9 & \Phi^{9}_{\hat{0}0} & \alpha_{10} & \Phi^{2}_{\downarrow 0} & \Phi^{6}_{\downarrow 0} & \Phi^{9}_{\downarrow 0} & \Phi^{4}_{\downarrow 0} \\
\Phi^{9}_{\downarrow\uparrow} & \Phi^{10}_{\downarrow\uparrow} & \Phi^{11}_{\downarrow\uparrow} & \Phi^{15}_{\downarrow\uparrow} & \Phi^{9}_{\downarrow\uparrow} & \Phi^{10}_{\downarrow\downarrow} & \Phi^{11}_{\downarrow\downarrow} & \Phi^{15}_{\downarrow\downarrow} & \alpha_6 & \alpha_8 & \alpha_7 & \Phi^{10}_{\hat{0}0} & \Phi^{3}_{\downarrow 0} & \Phi^{5}_{\downarrow 0} & \Phi^{4}_{\downarrow 0} & \Phi^{10}_{\downarrow 0} \\
\lambda_4 & \Phi^{4}_{\uparrow\uparrow} & \Phi^{5}_{\uparrow\uparrow} & \Phi^{12}_{\uparrow\uparrow} & \lambda_{10} & \Phi^{4}_{\uparrow\downarrow} & \Phi^{5}_{\uparrow\downarrow} & \Phi^{12}_{\uparrow\downarrow} & \Phi^{7}_{\uparrow 0} & \Phi^{1}_{\uparrow 0} & \Phi^{2}_{\uparrow 0} & \Phi^{3}_{\uparrow 0} & -\Phi^{7}_{\hat{0}0} & -\alpha_4 & -\alpha_5 & -\alpha_6 \\
\Phi^{1}_{\uparrow\uparrow} & \lambda_5 & \Phi^{6}_{\uparrow\uparrow} & \Phi^{13}_{\uparrow\uparrow} & \Phi^{1}_{\uparrow\downarrow} & \lambda_{11} & \Phi^{6}_{\uparrow\downarrow} & \Phi^{13}_{\uparrow\downarrow} & \Phi^{1}_{\uparrow 0} & \Phi^{8}_{\uparrow 0} & \Phi^{6}_{\uparrow 0} & \Phi^{5}_{\uparrow 0} & -\alpha_1 & -\Phi^{8}_{\hat{0}0} & -\alpha_9 & -\alpha_8 \\
\Phi^{2}_{\uparrow\uparrow} & \Phi^{3}_{\uparrow\uparrow} & \lambda_6 & \Phi^{14}_{\uparrow\uparrow} & \Phi^{2}_{\uparrow\downarrow} & \Phi^{3}_{\uparrow\downarrow} & \lambda_{12} & \Phi^{14}_{\uparrow\downarrow} & \Phi^{2}_{\uparrow 0} & \Phi^{6}_{\uparrow 0} & \Phi^{9}_{\uparrow 0} & \Phi^{4}_{\uparrow 0} & -\alpha_2 & -\alpha_{12} & -\Phi^{9}_{\hat{0}0} & -\alpha_7 \\
\Phi^{9}_{\uparrow\uparrow} & \Phi^{10}_{\uparrow\uparrow} & \Phi^{11}_{\uparrow\uparrow} & \Phi^{15}_{\uparrow\uparrow} & \Phi^{9}_{\uparrow\downarrow} & \Phi^{10}_{\uparrow\downarrow} & \Phi^{11}_{\uparrow\downarrow} & \Phi^{15}_{\uparrow\downarrow} & \Phi^{3}_{\uparrow 0} & \Phi^{5}_{\uparrow 0} & \Phi^{4}_{\uparrow 0} & \Phi^{10}_{\uparrow 0} & -\alpha_3 & -\alpha_{11} & -\alpha_{10} & -\Phi^{10}_{\hat{0}0} \\
\Phi^{7}_{0\hat{0}} & \beta_1 & \beta_2 & \beta_3 & \Phi^{7}_{0\downarrow} & \Phi^{1}_{0\downarrow} & \Phi^{2}_{0\downarrow} & \Phi^{3}_{0\downarrow} & \lambda_{10} & \Phi^{1}_{\uparrow\downarrow} & \Phi^{2}_{\uparrow\downarrow} & \Phi^{9}_{\uparrow\downarrow} & -\lambda_7 & \Phi^{1}_{\downarrow\downarrow} & \Phi^{2}_{\downarrow\downarrow} & \Phi^{9}_{\downarrow\downarrow} \\
\beta_4 & \Phi^{8}_{0\hat{0}} & \beta_{12} & \beta_{11} & \Phi^{1}_{0\downarrow} & \Phi^{8}_{0\downarrow} & \Phi^{6}_{0\downarrow} & \Phi^{5}_{0\downarrow} & \Phi^{4}_{\uparrow\downarrow} & \lambda_{11} & \Phi^{3}_{\uparrow\downarrow} & \Phi^{10}_{\uparrow\downarrow} & \Phi^{4}_{\downarrow\downarrow} & -\lambda_8 & \Phi^{3}_{\downarrow\downarrow} & \Phi^{10}_{\downarrow\downarrow} \\
\beta_5 & \beta_9 & \Phi^{9}_{0\hat{0}} & \beta_{10} & \Phi^{2}_{0\downarrow} & \Phi^{6}_{0\downarrow} & \Phi^{9}_{0\downarrow} & \Phi^{4}_{0\downarrow} & \Phi^{5}_{\uparrow\downarrow} & \Phi^{6}_{\uparrow\downarrow} & \lambda_{12} & \Phi^{11}_{\uparrow\downarrow} & \Phi^{5}_{\downarrow\downarrow} & \Phi^{6}_{\downarrow\downarrow} & -\lambda_9 & \Phi^{11}_{\downarrow\downarrow} \\
\beta_6 & \beta_8 & \beta_7 & \Phi^{10}_{0\hat{0}} & \Phi^{3}_{0\downarrow} & \Phi^{5}_{0\downarrow} & \Phi^{4}_{0\downarrow} & \Phi^{10}_{0\downarrow} & \Phi^{12}_{\uparrow\downarrow} & \Phi^{13}_{\uparrow\downarrow} & \Phi^{14}_{\uparrow\downarrow} & \Phi^{15}_{\uparrow\downarrow} & \Phi^{12}_{\downarrow\downarrow} & \Phi^{13}_{\downarrow\downarrow} & \Phi^{14}_{\downarrow\downarrow} & \Phi^{15}_{\downarrow\downarrow} \\
\Phi^{7}_{0\uparrow} & \Phi^{1}_{0\uparrow} & \Phi^{2}_{0\uparrow} & \Phi^{3}_{0\uparrow} & -\Phi^{7}_{0\hat{0}} & -\beta_4 & -\beta_5 & -\beta_6 & -\lambda_4 & \Phi^{1}_{\uparrow\uparrow} & \Phi^{2}_{\uparrow\uparrow} & \Phi^{9}_{\uparrow\uparrow} & \lambda_1 & \Phi^{1}_{\downarrow\uparrow} & \Phi^{2}_{\downarrow\uparrow} & \Phi^{9}_{\downarrow\uparrow} \\
\Phi^{1}_{0\uparrow} & \Phi^{8}_{0\uparrow} & \Phi^{6}_{0\uparrow} & \Phi^{5}_{0\uparrow} & -\beta_1 & -\Phi^{8}_{0\hat{0}} & -\beta_9 & -\beta_8 & \Phi^{4}_{\uparrow\uparrow} & -\lambda_5 & \Phi^{3}_{\uparrow\uparrow} & \Phi^{10}_{\uparrow\uparrow} & \Phi^{4}_{\downarrow\uparrow} & \lambda_2 & \Phi^{3}_{\downarrow\uparrow} & \Phi^{10}_{\downarrow\uparrow} \\
\Phi^{2}_{0\uparrow} & \Phi^{6}_{0\uparrow} & \Phi^{9}_{0\uparrow} & \Phi^{4}_{0\uparrow} & -\beta_2 & -\beta_{12} & -\Phi^{9}_{0\hat{0}} & -\beta_7 & \Phi^{5}_{\uparrow\uparrow} & \Phi^{6}_{\uparrow\uparrow} & -\lambda_6 & \Phi^{11}_{\uparrow\uparrow} & \Phi^{5}_{\downarrow\uparrow} & \Phi^{6}_{\downarrow\uparrow} & \lambda_3 & \Phi^{11}_{\downarrow\uparrow} \\
\Phi^{3}_{0\uparrow} & \Phi^{5}_{0\uparrow} & \Phi^{4}_{0\uparrow} & \Phi^{10}_{0\uparrow} & -\beta_3 & -\beta_{11} & -\beta_{10} & -\Phi^{10}_{0\hat{0}} & \Phi^{12}_{\uparrow\uparrow} & \Phi^{13}_{\uparrow\uparrow} & \Phi^{14}_{\uparrow\uparrow} & \Phi^{15}_{\uparrow\uparrow} & \Phi^{12}_{\downarrow\uparrow} & \Phi^{13}_{\downarrow\uparrow} & \Phi^{14}_{\downarrow\uparrow} & \Phi^{15}_{\downarrow\uparrow}
\end{array}\right]$$

In this block, all positions are occupied and no Higgs field is charge conjugated to any other in the *same* block. On the other side $\Omega_{12} = \Omega_{21}^\dagger$, so that the lower block is occupied by the $\overline{126}$ Higgs fields. The 126 decomposes under the $SU(2)_L \times SU(2)_R \times SU(4)_c$ symmetry into the $(3, 1, \bar{10}) \oplus (1, 3, 10) \oplus (1, 1, 6) \oplus (2, 2, 15)$ multiplets [53]. Initially we will consider the first three multiplets and later on the last one. Let us first introduce explicitly the

Higgs fields which occupy the same cites in the above matrix. These entries are shown with β_i, α_i and λ_i. In the Higgs field matrix above, the entries β_i and α_i which are defined for $i = (1, \ldots, 12)$ are composed of the following fields

$$\begin{array}{llll} \beta_1 = \Phi^1_{0\hat{0}} - \Phi^1_{00} & \beta_7 = \Phi^4_{0\hat{0}} + \Phi^4_{00} & \alpha_1 = \Phi^1_{\hat{0}0} + \Phi^4_{00} & \alpha_7 = \Phi^4_{\hat{0}0} - \Phi^1_{00} \\ \beta_2 = \Phi^2_{0\hat{0}} - \Phi^2_{00} & \beta_8 = \Phi^5_{0\hat{0}} - \Phi^5_{00} & \alpha_2 = \Phi^2_{\hat{0}0} + \Phi^5_{00} & \alpha_8 = \Phi^5_{\hat{0}0} + \Phi^2_{00} \\ \beta_3 = \Phi^3_{0\hat{0}} - \Phi^3_{00} & \beta_9 = \Phi^6_{0\hat{0}} + \Phi^6_{00} & \alpha_3 = \Phi^3_{\hat{0}0} + \Phi^6_{00} & \alpha_9 = \Phi^6_{\hat{0}0} - \Phi^3_{00} \\ \beta_4 = \Phi^1_{0\hat{0}} + \Phi^1_{00} & \beta_{10} = \Phi^4_{0\hat{0}} - \Phi^4_{00} & \alpha_4 = \Phi^1_{\hat{0}0} - \Phi^4_{00} & \alpha_{10} = \Phi^4_{\hat{0}0} + \Phi^1_{00} \\ \beta_5 = \Phi^2_{0\hat{0}} + \Phi^2_{00} & \beta_{11} = \Phi^5_{0\hat{0}} + \Phi^5_{00} & \alpha_5 = \Phi^2_{\hat{0}0} - \Phi^5_{00} & \alpha_{11} = \Phi^5_{\hat{0}0} - \Phi^2_{00} \\ \beta_6 = \Phi^3_{0\hat{0}} + \Phi^3_{00} & \beta_{12} = \Phi^6_{0\hat{0}} - \Phi^6_{00} & \alpha_6 = \Phi^3_{\hat{0}0} - \Phi^6_{00} & \alpha_{12} = \Phi^6_{\hat{0}0} + \Phi^3_{00} \end{array} \tag{7.2}$$

In the above expressions Φ^i_{00} are left and right isospin singlet Higgs fields. This is implied by the two zeros in the subscript. There are totally six such Higgs fields and they form a sextet $(1, 1, 6)$ for which $i = (1, \ldots, 6)$. These fields are listed in the first column in eq. (7.3). On the other side we have ten $\Phi^i_{\hat{0}0}$ and ten $\Phi^i_{0\hat{0}}$ fields which have the same $SU(4)$ weights with the fields of the sextet for $i = (1, \ldots 6)$. Actually they belong to the $(3, 1, \bar{10})$ and $(1, 3, 10)$ multiplets respectively and exhibit a triplet structure with respect to left and right isospin groups respectively and do sit at the same sites with $(1, 1, 6)$ in the Higgs matrix. To avoid any inconsistent labelling, we have put a hat on the zeros wherever the zero denotes a zero of the respective isospin triplet. Indeed the $\Phi^i_{\hat{0}0}$ and $\Phi^i_{0\hat{0}}$ fields belong to separate decouplets where i runs up to 10. The other partners of the above mentioned six fields are the $\Phi^7_{0\hat{0}}$, $\Phi^8_{0\hat{0}}$, $\Phi^9_{0\hat{0}}$,$\Phi^{10}_{0\hat{0}}$ and the $\Phi^7_{\hat{0}0}$, $\Phi^8_{\hat{0}0}$, $\Phi^9_{\hat{0}0}$, $\Phi^{10}_{\hat{0}0}$ fields respectively and are seen to lie along the diagonal in the same respective blocks. These fields are listed in the second and in the last column in eq. (7.3) respectively where $\Delta \equiv (\uparrow, \hat{0}, \downarrow)$ denotes the triplet structure of the respective isospin gauge group. We have

$$\begin{array}{lll} & \Phi^1_{(\Delta,0,\,0,-\frac{1}{\sqrt{3}},-\frac{1}{\sqrt{6}})} & \Phi^1_{(0,\Delta,\,0,+\frac{1}{\sqrt{3}},+\frac{1}{\sqrt{6}})} \\ & \Phi^2_{(\Delta,0,-\frac{1}{2},+\frac{1}{2\sqrt{3}},-\frac{1}{\sqrt{6}})} & \Phi^2_{(0,\Delta,+\frac{1}{2},-\frac{1}{2\sqrt{3}},+\frac{1}{\sqrt{6}})} \\ \Phi^1_{(0,0,\,0\,,+\frac{1}{\sqrt{3}}\,,+\frac{1}{\sqrt{6}})} & \Phi^3_{(\Delta,0,-\frac{1}{2},-\frac{1}{2\sqrt{3}},+\frac{1}{\sqrt{6}})} & \Phi^3_{(0,\Delta,+\frac{1}{2},+\frac{1}{2\sqrt{3}},-\frac{1}{\sqrt{6}})} \\ \Phi^2_{(0,0,+\frac{1}{2},-\frac{1}{2\sqrt{3}},+\frac{1}{\sqrt{6}})} & \Phi^4_{(\Delta,0,\,0,+\frac{1}{\sqrt{3}},+\frac{1}{\sqrt{6}})} & \Phi^4_{(0,\Delta,\,0,-\frac{1}{\sqrt{3}},-\frac{1}{\sqrt{6}})} \\ \Phi^3_{(0,0,+\frac{1}{2},+\frac{1}{2\sqrt{3}},-\frac{1}{\sqrt{6}})} & \Phi^5_{(\Delta,0,+\frac{1}{2},-\frac{1}{2\sqrt{3}},+\frac{1}{\sqrt{6}})} & \Phi^5_{(0,\Delta,-\frac{1}{2},+\frac{1}{2\sqrt{3}},-\frac{1}{\sqrt{6}})} \\ \Phi^4_{(0,0,\,0\,,-\frac{1}{\sqrt{3}}\,,-\frac{1}{\sqrt{6}})} & \Phi^6_{(\Delta,0,+\frac{1}{2},+\frac{1}{2\sqrt{3}},-\frac{1}{\sqrt{6}})} & \Phi^6_{(0,\Delta,-\frac{1}{2},-\frac{1}{2\sqrt{3}},+\frac{1}{\sqrt{6}})} \\ \Phi^5_{(0,0,-\frac{1}{2},+\frac{1}{2\sqrt{3}},-\frac{1}{\sqrt{6}})} & \Phi^7_{(\Delta,0,-1,-\frac{1}{\sqrt{3}},-\frac{1}{\sqrt{6}})} & \Phi^7_{(0,\Delta,+1,+\frac{1}{\sqrt{3}},+\frac{1}{\sqrt{6}})} \\ \Phi^6_{(0,0,-\frac{1}{2},-\frac{1}{2\sqrt{3}},+\frac{1}{\sqrt{6}})} & \Phi^8_{(\Delta,0,+1,-\frac{1}{\sqrt{3}},-\frac{1}{\sqrt{6}})} & \Phi^8_{(0,\Delta,-1,+\frac{1}{\sqrt{3}},+\frac{1}{\sqrt{6}})} \\ & \Phi^9_{(\Delta,0,\,0,+\frac{2}{\sqrt{3}},-\frac{1}{\sqrt{6}})} & \Phi^9_{(0,\Delta,\,0,-\frac{2}{\sqrt{3}},+\frac{1}{\sqrt{6}})} \\ & \Phi^{10}_{(\Delta,0,\,0,\,0\,,+\frac{3}{\sqrt{6}})} & \Phi^{10}_{(0,\Delta,\,0,\,0\,,-\frac{3}{\sqrt{6}})} \end{array} \tag{7.3}$$

Other elements of the $(3, 1, \bar{10})$ and $(1, 3, 10)$ multiplets occupy the 4×4 blocks positioned along the off-diagonals and can be easily recognized from their labels. In the Higgs matrix, the isospin triplet fields are symbolically labelled with $(\uparrow, \hat{0}, \downarrow)$ which indeed correspond to the $(+1, 0, -1)$ values of the weights. Other weights of these fields are in the Higgs matrix suppressed, but are given in the parentheses in equation eq. (7.3). These values are $SU(2)_L \times SU(2)_R \times SU(4)$ weights in the order $(L_3, R_3, U_3, U_8, U_{15})$ respectively. The weight diagrams of the 126 are given in Fig. (7.3). The explicit expression of the charge raised fields Φ in eq. (7.3) and the $\mathbf{\Gamma}$'s in terms of ϕ_{ijklm} and $\Gamma_i\Gamma_j\Gamma_k\Gamma_l\Gamma_m$ respectively are listed in appendix C.

Since all the sextet fields in eq. (7.3) posses color and electric charge, they are no good for symmetry breaking and can be excluded in advance. These fields decompose under $SU(2)_L \times SU(2)_R \times SU(3) \times U(1)_{B-L}$ into the $(1, 1, 3)$ and the $(1, 1, \bar{3})$ color triplets which differ by the $B - L$ numbers and the members of these triplets carry $-1/3$ and $1/3$ fractional electric charge respectively.

The only fields which are $SU(3)$ singlets in the $(3, 1, \bar{10})$ and $(1, 3, 10)$ multiplets are the 10^{th} fields. The 10^{th} fields in eq. (7.3) decompose under $SU(2)_L \times SU(2)_R \times SU(3) \times U(1)_{B-L}$ into the $(3, 1, \bar{1})$ and $(1, 3, 1)$ multiplets as illustrated in Fig. (7.3). The electric charges as well as other charges of these 10^{th} fields are listed in Table (7.1). Since electric-charge conservation should not be allowed to be violated, only the two electrically neutral components of the $(3, 1, \bar{1})$ and $(1, 3, 1)$ color singlet fields are good for symmetry breaking. The neutral components are $U(1)_Q$ singlets. We have

$$\begin{array}{ll} (i)\ \mathbf{\Gamma}^{10}_{\downarrow 0} \equiv \mathbf{\Gamma}^{10}_{(-1,0,\,0,\,0\,,+\frac{3}{\sqrt{6}})} \to \Delta_L + \dfrac{1}{2}\Delta_{B-L} = 0 = \Delta_Q, & \Delta_R = 0 \\ (ii)\ \mathbf{\Gamma}^{10}_{0\uparrow} \equiv \mathbf{\Gamma}^{10}_{(0,+1,\,0,\,0\,,-\frac{3}{\sqrt{6}})} \to \Delta_R + \dfrac{1}{2}\Delta_{B-L} = 0 = \Delta_Q = \Delta_Y, & \Delta_L = 0 \end{array} \tag{7.4}$$

where the Δ's in the upper line indicate the amount of non-commutation of $\mathbf{\Gamma}^{10}_{\downarrow 0}$ with the diagonal generator of the respective symmetry and the Δ's in the lower line indicate the amount of non-commutation of $\mathbf{\Gamma}^{10}_{0\uparrow}$ with the diagonal generator of the respective symmetry. The factor $1/2$ is due to the factor in the electric charge eigenvalue operator, namely $Q = L + R + (B-L)/2$ and $(B-L) = 2\sqrt{2/3}\, U_{15}$. Let us consider the following terms qualifying for the Higgs mechanism

$$\Phi_{31\bar{1}\bar{0}} + \Phi_{1310} \equiv \frac{1}{32}\mathbf{\Gamma}^{10}_{\downarrow 0}\,\Phi^{10}_{\downarrow 0} + \frac{1}{32}\mathbf{\Gamma}^{10}_{0\uparrow}\,\Phi^{10}_{0\uparrow}\ ;\quad Tr\left[(\Phi_{31\bar{1}\bar{0}})^2\right] = (\Phi^{10}_{\downarrow 0})^2,\quad Tr\left[(\Phi_{1310})^2\right] = (\Phi^{10}_{0\uparrow})^2 \tag{7.5}$$

where $(\ )^2 = (\ )(\ )^\dagger$. From now on, we will always refer to the above two Higgs fields on the right hand side with $(3,1,\bar{1}\bar{0})$ and $(1,3,10)$ respectively, or equivalently with $\Phi_{31\bar{1}\bar{0}}$ and Φ_{1310} respectively. $\mathbf{\Gamma}^{10}_{\downarrow 0}$ in terms of the Γ_i basis is expressible as

$$\begin{aligned}
\mathbf{\Gamma}^{10}_{\downarrow 0} = &-\Gamma_1\Gamma_3\Gamma_5\Gamma_7\Gamma_{10} + i\Gamma_1\Gamma_3\Gamma_5\Gamma_7\Gamma_9 - i\Gamma_1\Gamma_3\Gamma_5\Gamma_8\Gamma_{10} - \Gamma_1\Gamma_3\Gamma_5\Gamma_8\Gamma_9\\
&-i\Gamma_1\Gamma_3\Gamma_6\Gamma_7\Gamma_{10} - \Gamma_1\Gamma_3\Gamma_6\Gamma_7\Gamma_9 + \Gamma_1\Gamma_3\Gamma_6\Gamma_8\Gamma_{10} - i\Gamma_1\Gamma_3\Gamma_6\Gamma_8\Gamma_9\\
&+i\Gamma_1\Gamma_4\Gamma_5\Gamma_7\Gamma_{10} + \Gamma_1\Gamma_4\Gamma_5\Gamma_7\Gamma_9 - \Gamma_1\Gamma_4\Gamma_5\Gamma_8\Gamma_{10} + i\Gamma_1\Gamma_4\Gamma_5\Gamma_8\Gamma_9\\
&-\Gamma_1\Gamma_4\Gamma_6\Gamma_7\Gamma_{10} + i\Gamma_1\Gamma_4\Gamma_6\Gamma_7\Gamma_9 - i\Gamma_1\Gamma_4\Gamma_6\Gamma_8\Gamma_{10} - \Gamma_1\Gamma_4\Gamma_6\Gamma_8\Gamma_9\\
&+i\Gamma_2\Gamma_4\Gamma_6\Gamma_8\Gamma_9 - \Gamma_2\Gamma_4\Gamma_6\Gamma_8\Gamma_{10} + \Gamma_2\Gamma_4\Gamma_6\Gamma_7\Gamma_9 + i\Gamma_2\Gamma_4\Gamma_6\Gamma_7\Gamma_{10}\\
&+\Gamma_2\Gamma_4\Gamma_5\Gamma_8\Gamma_9 + i\Gamma_2\Gamma_4\Gamma_5\Gamma_8\Gamma_{10} - i\Gamma_2\Gamma_4\Gamma_5\Gamma_7\Gamma_9 + \Gamma_2\Gamma_4\Gamma_5\Gamma_7\Gamma_{10}\\
&-\Gamma_2\Gamma_3\Gamma_6\Gamma_8\Gamma_9 - i\Gamma_2\Gamma_3\Gamma_6\Gamma_8\Gamma_{10} + i\Gamma_2\Gamma_3\Gamma_6\Gamma_7\Gamma_9 - \Gamma_2\Gamma_3\Gamma_6\Gamma_7\Gamma_{10}\\
&+i\Gamma_2\Gamma_3\Gamma_5\Gamma_8\Gamma_9 - \Gamma_2\Gamma_3\Gamma_5\Gamma_8\Gamma_{10} + \Gamma_2\Gamma_3\Gamma_5\Gamma_7\Gamma_9 + i\Gamma_2\Gamma_3\Gamma_5\Gamma_7\Gamma_{10}
\end{aligned} \tag{7.6}$$

If we use the Γ_i matrices of basis C, the $\mathbf{\Gamma}^{10}_{\downarrow 0}$ matrix turns out to have a non-vanishing entry i.e., 32 at position (4,32) where all other entries are zero. This is directly observable from the Higgs field matrix of the 126 as well. The position $(4,32)$ lies in the Ω_{12} block. If we use basis B to evaluate the matrix above than the non zero entry lies at position $(7,26)$ and for basis A, it lies at position $(2,21)$. The common property of these positions is that it correctly couples the Higgs field $\Phi^{10}_{\downarrow 0}$ to two neutrinos which are ν_L and $(\nu^c)_R$. This will be considered in some more detail in § 7.1. The $\mathbf{\Gamma}^{10}_{0\uparrow}$ in terms of the Γ_i basis is found as

$$\begin{aligned}
\mathbf{\Gamma}^{10}_{0\uparrow} = &+\Gamma_1\Gamma_3\Gamma_5\Gamma_7\Gamma_{10} - i\Gamma_1\Gamma_3\Gamma_5\Gamma_7\Gamma_9 - i\Gamma_1\Gamma_3\Gamma_5\Gamma_8\Gamma_{10} - \Gamma_1\Gamma_3\Gamma_5\Gamma_8\Gamma_9\\
&-i\Gamma_1\Gamma_3\Gamma_6\Gamma_7\Gamma_{10} - \Gamma_1\Gamma_3\Gamma_6\Gamma_7\Gamma_9 - \Gamma_1\Gamma_3\Gamma_6\Gamma_8\Gamma_{10} + i\Gamma_1\Gamma_3\Gamma_6\Gamma_8\Gamma_9\\
&+i\Gamma_1\Gamma_4\Gamma_5\Gamma_7\Gamma_{10} + \Gamma_1\Gamma_4\Gamma_5\Gamma_7\Gamma_9 + \Gamma_1\Gamma_4\Gamma_5\Gamma_8\Gamma_{10} - i\Gamma_1\Gamma_4\Gamma_5\Gamma_8\Gamma_9\\
&+\Gamma_1\Gamma_4\Gamma_6\Gamma_7\Gamma_{10} - i\Gamma_1\Gamma_4\Gamma_6\Gamma_7\Gamma_9 - i\Gamma_1\Gamma_4\Gamma_6\Gamma_8\Gamma_{10} - \Gamma_1\Gamma_4\Gamma_6\Gamma_8\Gamma_9\\
&-i\Gamma_2\Gamma_4\Gamma_6\Gamma_8\Gamma_9 + \Gamma_2\Gamma_4\Gamma_6\Gamma_8\Gamma_{10} + \Gamma_2\Gamma_4\Gamma_6\Gamma_7\Gamma_9 + i\Gamma_2\Gamma_4\Gamma_6\Gamma_7\Gamma_{10}\\
&+\Gamma_2\Gamma_4\Gamma_5\Gamma_8\Gamma_9 + i\Gamma_2\Gamma_4\Gamma_5\Gamma_8\Gamma_{10} + i\Gamma_2\Gamma_4\Gamma_5\Gamma_7\Gamma_9 - \Gamma_2\Gamma_4\Gamma_5\Gamma_7\Gamma_{10}\\
&-\Gamma_2\Gamma_3\Gamma_6\Gamma_8\Gamma_9 - i\Gamma_2\Gamma_3\Gamma_6\Gamma_8\Gamma_{10} - i\Gamma_2\Gamma_3\Gamma_6\Gamma_7\Gamma_9 + \Gamma_2\Gamma_3\Gamma_6\Gamma_7\Gamma_{10}\\
&-i\Gamma_2\Gamma_3\Gamma_5\Gamma_8\Gamma_9 + \Gamma_2\Gamma_3\Gamma_5\Gamma_8\Gamma_{10} + \Gamma_2\Gamma_3\Gamma_5\Gamma_7\Gamma_9 + i\Gamma_2\Gamma_3\Gamma_5\Gamma_7\Gamma_{10}
\end{aligned} \tag{7.7}$$

Again if we use the Γ_i matrices in basis C, then $\mathbf{\Gamma}^{10}_{0\uparrow}$ turns out to have a non-vanishing entry i.e., 32 at position (16,20) where all other entries are zero. Notice that $(16,20)$ lies in Ω_{12} block which is also directly observable from the Higgs field matrix of the 126. If we use basis B to evaluate the matrix above, the non zero entry lies at position $(18,15)$ and for basis A, it lies at position $(5,18)$. The common property of these positions is that it correctly couples the Higgs field $\Phi^{10}_{0\uparrow}$ to two neutrinos which are $(\nu^c)_L$ and ν_R. Notice also that in each of the expressions for $\mathbf{\Gamma}^{10}_{\downarrow 0}$ and $\mathbf{\Gamma}^{10}_{0\uparrow}$ the first four rows are related to the terms in the last four rows over $-i\Gamma_{five} = \Gamma_1\cdots\Gamma_{10}$ as was pointed out before. We have the following commutation relations

$$\begin{aligned}
\left[\mathbf{\Gamma}^{10}_{\downarrow 0}, L_3\right] &= -1\ \mathbf{\Gamma}^{10}_{\downarrow 0} & \left[\mathbf{\Gamma}^{10}_{0\uparrow}, L_3\right] &= 0\ \mathbf{\Gamma}^{10}_{0\uparrow}\\
\left[\mathbf{\Gamma}^{10}_{\downarrow 0}, R_3\right] &= 0\ \mathbf{\Gamma}^{10}_{\downarrow 0} & \left[\mathbf{\Gamma}^{10}_{0\uparrow}, R_3\right] &= +1\ \mathbf{\Gamma}^{10}_{0\uparrow}\\
\left[\mathbf{\Gamma}^{10}_{\downarrow 0}, U_3\right] &= 0\ \mathbf{\Gamma}^{10}_{\downarrow 0} & \left[\mathbf{\Gamma}^{10}_{0\uparrow}, U_3\right] &= 0\ \mathbf{\Gamma}^{10}_{0\uparrow}\\
\left[\mathbf{\Gamma}^{10}_{\downarrow 0}, U_8\right] &= 0\ \mathbf{\Gamma}^{10}_{\downarrow 0} & \left[\mathbf{\Gamma}^{10}_{0\uparrow}, U_8\right] &= 0\ \mathbf{\Gamma}^{10}_{0\uparrow}\\
\left[\mathbf{\Gamma}^{10}_{\downarrow 0}, U_{15}\right] &= +\tfrac{3}{\sqrt{6}}\ \mathbf{\Gamma}^{10}_{\downarrow 0} & \left[\mathbf{\Gamma}^{10}_{0\uparrow}, U_{15}\right] &= -\tfrac{3}{\sqrt{6}}\ \mathbf{\Gamma}^{10}_{0\uparrow}
\end{aligned} \tag{7.8}$$

Before we continue elaborating the remaining fields in the Higgs matrix, we will sort out various symmetry breaking chains that the above Higgs term given in eq. (7.5), namely the $(3,1,\bar{10})$ and $(1,3,10)$ can initiate. The first component of the Higgs term breaks the following initial symmetries to the final symmetries

(i) $SU(4)\times SU(2)_L\times SU(2)_R$ down to $SU(3)_c\times SU(2)_R\times U(1)_{Y'}$ where the $U(1)_{Y'}$ is an Abelian gauge group whose eigenvalue operator is $L_3+(B-L)/2$. The $U(1)_{Y'}$ symmetry operating on the 16_L (or 16_R) spinor of fermions is understood to yield the sum of the L_3 and $(B-L)/2$ eigenvalues.

(ii) $SU(3)\times SU(2)_L\times SU(2)_R\times U(1)_{B-L}$ down to $SU(3)\times SU(2)_R\times U(1)_{Y'}$ as above.

(iii) $SU(3)\times SU(2)_R\times U(1)_{Y'}$ remains unbroken

(iv) $SU(3)\times SU(2)_L\times U(1)_Y$ down to $SU(3)\times U(1)_Q$.

The second component of the Higgs term breaks the subsequent initial symmetries to the final symmetries

(i) $SU(4)\times SU(2)_L\times SU(2)_R$ down to $SU(3)\times SU(2)_L\times U(1)_Y$

(ii) $SU(3)\times SU(2)_L\times SU(2)_R\times U(1)_{B-L}$ down to $SU(3)\times SU(2)_L\times U(1)_Y$ as above.

(iii) $SU(3)\times SU(2)_L\times U(1)_Y$ remains unbroken.

(iv) $SU(3)\times SU(2)_R\times U(1)_{Y'}$ down to $SU(3)\times U(1)_Q$

It is remarkable to notice that the left-right symmetric 4-color gauge group $always$ breaks down to $SU(3)\times U(1)_Q$, if $both$ components of the $(3,1,\bar{10})\oplus(1,3,10)$ operate. This can easily be verified from above. Some other cases that we had previously explored for the $(2,2,1)$ of the 10 can be reconsidered in cooperation with $(3,1,\bar{10})\oplus(1,3,10)$. In this respect we take the final symmetries that had been reached via the $(2,2,1)$ as initial symmetries on which the $(3,1,\bar{10})\oplus(1,3,10)$ can act. Then the $(3,1,\bar{10})$ breaks the following symmetries to the final symmetries

(i) $SU(4)\times U(1)_{L+R}$ down to $SU(3)\times U(1)_Q$

(ii) $SU(3)\times U(1)_{L+R}\times U(1)_{B-L}$ down to $SU(3)\times U(1)_Q$

The second term, i.e., the $(1,3,10)$ breaks the following initial symmetries to the final symmetries

(i) $SU(4)\times U(1)_{L+R}$ down to $SU(3)\times U(1)_Q$

(ii) $SU(3)\times U(1)_{L+R}\times U(1)_{B-L}$ down to $SU(3)\times U(1)_Q$

So we have made the observation that the $(2,2,1)$ and the $(1,3,10)\oplus(3,1,\bar{10})$ are very well cooperating. This cooperation is easier to observe from the illustration in Fig. (11.1) where all the above cases as well as the ones in § 6 are illustrated. It is seen that the Higgs $trio$, i.e., the Φ_{221}, Φ_{1310} and $\Phi_{31\bar{10}}$ never pushes the symmetry in a $cul\ de\ sac$, i.e., the $SU(3)_c\times U(1)_Q$ is reachable regardless of the order of the Higgs multiplets employed. The obvious $calculational$ advantage of this observation is that one has the freedom to employ the above Higgs multiplets in the Higgs mechanism (a) without choosing $predetermined$ chains of symmetries, or (b) without bothering about the order of the respective vevs. But at the end, we must be able to ascribe the vevs their correct values, as far as physics is concerned. This will be achieved in § 15

7.1 Features of the $\Phi_{1310}\oplus\Phi_{31\bar{10}}$

Two important features that are worth discussing is how the $\Phi_{1310}\oplus\Phi_{31\bar{10}}$ multiplets behave in the Yukawa sector and in the Higgs mechanism.

(i) Again we consider the most general case in that we break the $SO(10)$ gauge group with the Higgs field $(\mathbf{\Gamma}^{10}_{\downarrow 0}\,\Phi^{10}_{\downarrow 0}+\mathbf{\Gamma}^{10}_{0\uparrow}\,\Phi^{10}_{0\uparrow})$. Then the resulting vev of the *first* term endows the following gauge boson of the 45 with mass : X_{B-L}, W^-_R, W^0_R, W^+_R, A_r, A_g, A_b , A'_r, A'_g, A'_b and X_r, X_g, X_b. The *second* term endows the following gauge boson of the 45 with mass : X_{B-L}, W^-_L, W^0_L, W^+_L, A_r, A_g, A_b , Y_r, Y_g, Y_b and X_r, X_g, X_b. The $remark$ in §6.2 applies here as well.

(ii) The first component of the above Higgs field induces mixing among $X_{B-L}-W^0_R$, where the sign $-$ is always illustrative. The mixed state has one massive and one massless mode. The second component of the above Higgs field gives rise to mixing among $X_{B-L}-W^0_L$. Similarly this mixed state has also one massive and one massless mode.

(iii) If both Higgs fields receive vevs there occurs a mixing among $W_L - W_R - X_{B-L}$ where one mode is massless and two modes are massive. The two massive modes are high- and low-mass modes.

(iv) The $\mathbf{\Gamma}^{10}_{\downarrow 0}\,\Phi^{10}_{\downarrow 0} + \mathbf{\Gamma}^{10}_{0\uparrow}\,\Phi^{10}_{0\uparrow}$ Higgs fields as pointed out previously couple to neutrinos as was shown in eqs. (7.6) and (7.7). Let us consider the Yukawa term $Y_{ij}\,\bar{\Psi}_i\,(\Phi_{1310} + \Phi_{31\overline{10}})\,\Psi_j$ where $i, j = (1,2,3)$ are family indices, Y_{ij} are the Yukawa couplings and Ψ_i are the family spinors. The vacuum expectation values of $\mathbf{\Gamma}^{10}_{\downarrow 0}\,\Phi^{10}_{\downarrow 0} + \mathbf{\Gamma}^{10}_{0\uparrow}\,\Phi^{10}_{0\uparrow}$ will endow the neutrinos with Majorana masses. These vertices are shown in Fig. (7.1). The charges conserved at both vertices can be checked from Tables (4.3), (4.4) and (7.1). These are Majorana mass terms like $\overline{\nu_L}\,(\nu_L)^c$ and $\overline{(\nu_R)^c}\,\nu_R$.

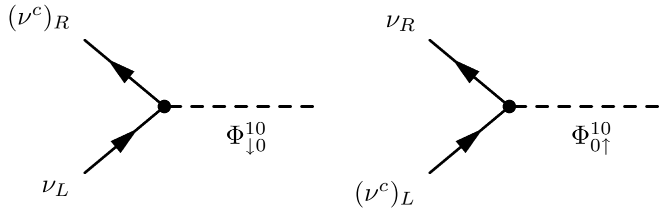

Fig. 7.1: The scalars $\Phi^{10}_{\downarrow 0}$ and $\Phi^{10}_{0\uparrow}$ coupling to neutrinos. The scalars are flowing into the vertices.

Details concerning the above features will be considered later in § 11 and § 12. Let us come back to those fields of the 126 which we haven't discussed yet. In the Higgs field matrix above, the entries λ_i along the diagonal for $i = (1, \ldots, 12)$ are multiply occupied. We have

$$\begin{array}{rclrcl}
\lambda_1 &=& \Phi^7_{\downarrow\uparrow} - \Phi^{15}_{\downarrow\uparrow}/3 & \lambda_7 &=& \Phi^7_{\downarrow\downarrow} - \Phi^{15}_{\downarrow\downarrow}/3 \\
\lambda_2 &=& \Phi^8_{\downarrow\uparrow} - \Phi^{15}_{\downarrow\uparrow}/3 & \lambda_8 &=& \Phi^7_{\downarrow\downarrow} - \Phi^{15}_{\downarrow\downarrow}/3 \\
\lambda_3 &=& -\Phi^7_{\downarrow\uparrow} - \Phi^8_{\downarrow\uparrow} - \Phi^{15}_{\downarrow\uparrow}/3 & \lambda_9 &=& -\Phi^7_{\downarrow\downarrow} - \Phi^8_{\downarrow\downarrow} - \Phi^{15}_{\downarrow\downarrow}/3 \\
\lambda_4 &=& \Phi^7_{\uparrow\uparrow} - \Phi^{15}_{\uparrow\uparrow}/3 & \lambda_{10} &=& \Phi^7_{\uparrow\downarrow} - \Phi^{15}_{\uparrow\downarrow}/3 \\
\lambda_5 &=& \Phi^8_{\uparrow\uparrow} - \Phi^{15}_{\uparrow\uparrow}/3 & \lambda_{11} &=& \Phi^8_{\uparrow\downarrow} - \Phi^{15}_{\uparrow\downarrow}/3 \\
\lambda_6 &=& -\Phi^7_{\uparrow\uparrow} - \Phi^8_{\uparrow\uparrow} - \Phi^{15}_{\uparrow\uparrow}/3 & \lambda_{12} &=& -\Phi^7_{\uparrow\downarrow} - \Phi^8_{\uparrow\downarrow} - \Phi^{15}_{\uparrow\downarrow}/3
\end{array} \tag{7.9}$$

These fields are part of a bi-doublet in the $SU(2)_L \times SU(2)_R$ space and simultaneously satisfy a 15-plet in the $SU(4)$ space. We labelled each field of this $(2,2,15)$ multiplet with an upper index i where $i = (1, \ldots 15)$. There are totally 60 Higgs fields falling in this multiplet, and they are listed together with their weights as below.

$$\begin{array}{ll}
\Phi^1_{(\pm\frac{1}{2},\pm\frac{1}{2},+1,0,0)} & \Phi^1_{(\pm\frac{1}{2},\mp\frac{1}{2},+1,0,0)} \\
\Phi^2_{(\pm\frac{1}{2},\pm\frac{1}{2},+\frac{1}{2},+\frac{3}{2\sqrt{3}},0)} & \Phi^2_{(\pm\frac{1}{2},\mp\frac{1}{2},+\frac{1}{2},+\frac{3}{2\sqrt{3}},0)} \\
\Phi^3_{(\pm\frac{1}{2},\pm\frac{1}{2},-\frac{1}{2},+\frac{3}{2\sqrt{3}},0)} & \Phi^3_{(\pm\frac{1}{2},\mp\frac{1}{2},-\frac{1}{2},+\frac{3}{2\sqrt{3}},0)} \\
\Phi^4_{(\pm\frac{1}{2},\pm\frac{1}{2},-1,0,0)} & \Phi^4_{(\pm\frac{1}{2},\mp\frac{1}{2},-1,0,0)} \\
\Phi^5_{(\pm\frac{1}{2},\pm\frac{1}{2},-\frac{1}{2},-\frac{3}{2\sqrt{3}},0)} & \Phi^5_{(\pm\frac{1}{2},\mp\frac{1}{2},-\frac{1}{2},-\frac{3}{2\sqrt{3}},0)} \\
\Phi^6_{(\pm\frac{1}{2},\pm\frac{1}{2},+\frac{1}{2},-\frac{3}{2\sqrt{3}},0)} & \Phi^6_{(\pm\frac{1}{2},\mp\frac{1}{2},+\frac{1}{2},-\frac{3}{2\sqrt{3}},0)} \\
\Phi^7_{(\pm\frac{1}{2},\pm\frac{1}{2},0,0,0)} & \Phi^7_{(\pm\frac{1}{2},\mp\frac{1}{2},0,0,0)} \\
\Phi^8_{(\pm\frac{1}{2},\pm\frac{1}{2},0,0,0)} & \Phi^8_{(\pm\frac{1}{2},\mp\frac{1}{2},0,0,0)} \\
\Phi^9_{(\pm\frac{1}{2},\pm\frac{1}{2},+\frac{1}{2},+\frac{1}{2\sqrt{3}},+\frac{2}{\sqrt{6}})} & \Phi^9_{(\pm\frac{1}{2},\mp\frac{1}{2},+\frac{1}{2},+\frac{1}{2\sqrt{3}},+\frac{2}{\sqrt{6}})} \\
\Phi^{10}_{(\pm\frac{1}{2},\pm\frac{1}{2},-\frac{1}{2},+\frac{1}{2\sqrt{3}},+\frac{2}{\sqrt{6}})} & \Phi^{10}_{(\pm\frac{1}{2},\mp\frac{1}{2},-\frac{1}{2},+\frac{1}{2\sqrt{3}},+\frac{2}{\sqrt{6}})} \\
\Phi^{11}_{(\pm\frac{1}{2},\pm\frac{1}{2},0,-\frac{1}{\sqrt{3}},+\frac{2}{\sqrt{6}})} & \Phi^{11}_{(\pm\frac{1}{2},\mp\frac{1}{2},0,-\frac{1}{\sqrt{3}},+\frac{2}{\sqrt{6}})} \\
\Phi^{12}_{(\pm\frac{1}{2},\pm\frac{1}{2},-\frac{1}{2},-\frac{1}{2\sqrt{3}},-\frac{2}{\sqrt{6}})} & \Phi^{12}_{(\pm\frac{1}{2},\mp\frac{1}{2},-\frac{1}{2},-\frac{1}{2\sqrt{3}},-\frac{2}{\sqrt{6}})} \\
\Phi^{13}_{(\pm\frac{1}{2},\pm\frac{1}{2},+\frac{1}{2},-\frac{1}{2\sqrt{3}},-\frac{2}{\sqrt{6}})} & \Phi^{13}_{(\pm\frac{1}{2},\mp\frac{1}{2},+\frac{1}{2},-\frac{1}{2\sqrt{3}},-\frac{2}{\sqrt{6}})} \\
\Phi^{14}_{(\pm\frac{1}{2},\pm\frac{1}{2},0,+\frac{1}{\sqrt{3}},-\frac{2}{\sqrt{6}})} & \Phi^{14}_{(\pm\frac{1}{2},\mp\frac{1}{2},0,+\frac{1}{\sqrt{3}},-\frac{2}{\sqrt{6}})} \\
\Phi^{15}_{(\pm\frac{1}{2},\pm\frac{1}{2},0,0,0)} & \Phi^{15}_{(\pm\frac{1}{2},\mp\frac{1}{2},0,0,0)}
\end{array} \tag{7.10}$$

where in each weight configuration either up signs or down signs should be chosen. The weight diagrams of the $(1,1,15)$ multiplet is given in Fig. (7.3). In the Higgs field matrix the $(\pm 1/2)$ isospin states are symbolically indicated with 4 possible configurations $\uparrow\uparrow$, $\uparrow\downarrow$, $\downarrow\uparrow$ and $\downarrow\downarrow$ as subscripts. The remaining weights are suppressed.

But since there is always an upper index addressed for each field in the Higgs field matrix, it is possible to read off its remaining weights from the above given list.

Again the useful Higgs fields for implementing in the Higgs mechanism must be first of all $SU(3)_c$ singlets. As shown in Fig. (7.3), under the $SU(2)_L \times SU(2)_R \times SU(3)_c \times U(1)_{B-L}$ symmetry, the first eight bi-doublets above constitute a color octet, namely the $(2,2,8)$ and the next six bi-doublets decompose into two color triplets, which are the $(2,2,3)$ and the $(2,2,\bar{3})$ triplets. These carry all fractional electric charges. The last field Φ^{15} which comes with 4 possible left-right isospin configurations are $SU(3)_c$ singlets. Under $SU(2)_L \times SU(2)_R \times SU(3)_c \times U(1)_{B-L}$ these 4 fields decompose into the $(2,2,1)$ multiplet all with zero $B-L$ number. Various charges of these 4 fields are given in Table (7.1). Only the $\Phi^{15}_{\downarrow\uparrow}$ and $\Phi^{15}_{\uparrow\downarrow}$ Higgs fields are electrically neutral. It should be noted that the $(2,2,1)$ of the 10 and the above $(2,2,1)$ of the 126, are basically different in that the latter endows the X_α bosons with mass and the former not. This and related properties of the $(2,2,15)$ will be reconsidered later in §7.2. We have

$$\begin{aligned} (i)\ \mathbf{\Gamma}^{15}_{\downarrow\uparrow} &\equiv \mathbf{\Gamma}_{(+\frac{1}{2},-\frac{1}{2},0,0,0)} \to \Delta_L + \Delta_R = 0 = \Delta_Q, \quad \Delta_{B-L} = 0 \\ (ii)\ \mathbf{\Gamma}_{\uparrow\downarrow} &\equiv \mathbf{\Gamma}^{15}_{(-\frac{1}{2},+\frac{1}{2},0,0,0)} \to \Delta_R + \Delta_L = 0 = \Delta_Q, \quad \Delta_{B-L} = 0 \end{aligned} \tag{7.11}$$

where the Δ's in the upper line indicate the amount of non-commutation of $\mathbf{\Gamma}^{15}_{\downarrow\uparrow}$ with L_3 and R_3. The Δ's in the lower line indicate the amount of non-commutation of $\mathbf{\Gamma}^{15}_{\uparrow\downarrow}$ with L_3 and R_3. Let us consider the following term as qualifying for the Higgs potential

$$\Phi_{2215} \equiv \frac{1}{\sqrt{384}} \left(\mathbf{\Gamma}^{15}_{\downarrow\uparrow}\ \Phi^{15}_{\downarrow\uparrow} + \mathbf{\Gamma}^{15}_{\uparrow\downarrow}\ \Phi^{15}_{\uparrow\downarrow} \right); \qquad Tr[(\Phi_{2215})^2] = (\Phi^{15}_{\downarrow\uparrow})^2 + (\Phi^{15}_{\uparrow\downarrow})^2 \tag{7.12}$$

where $(\ )^2 = (\ )(\ )^\dagger$. From now on, we will always refer to the above two Higgs fields with the $(2,2,15)$ or equivalently with Φ_{2215}. In terms of the Γ_i matrices, $\mathbf{\Gamma}^{15}_{\uparrow\downarrow}$ and $\mathbf{\Gamma}^{15}_{\downarrow\uparrow}$ are given as

$$\begin{aligned} -\mathbf{\Gamma}^{15}_{\uparrow\downarrow} = \ (& -i\,\Gamma_1\Gamma_2\Gamma_3\Gamma_4\Gamma_{10} - \Gamma_1\Gamma_2\Gamma_3\Gamma_4\Gamma_9 + i\,\Gamma_1\Gamma_2\Gamma_5\Gamma_6\Gamma_{10} \\ & +\Gamma_1\Gamma_2\Gamma_5\Gamma_6\Gamma_9 + \ i\,\Gamma_1\Gamma_2\Gamma_7\Gamma_8\Gamma_{10} + \Gamma_1\Gamma_2\Gamma_7\Gamma_8\Gamma_9 \\ & -\Gamma_5\Gamma_6\Gamma_7\Gamma_8\Gamma_9 - \ i\,\Gamma_5\Gamma_6\Gamma_7\Gamma_8\Gamma_{10} + \Gamma_3\Gamma_4\Gamma_7\Gamma_8\Gamma_9 \\ & +i\,\Gamma_3\Gamma_4\Gamma_7\Gamma_8\Gamma_{10} + \Gamma_3\Gamma_4\Gamma_5\Gamma_6\Gamma_9 + i\,\Gamma_3\Gamma_4\Gamma_5\Gamma_6\Gamma_{10}\ ) \end{aligned} \tag{7.13}$$

$$\begin{aligned} -\mathbf{\Gamma}^{15}_{\downarrow\uparrow} = \ (& -i\,\Gamma_1\Gamma_2\Gamma_3\Gamma_4\Gamma_{10} + \Gamma_1\Gamma_2\Gamma_3\Gamma_4\Gamma_9 + i\Gamma_1\Gamma_2\Gamma_5\Gamma_6\Gamma_{10} \\ & -\Gamma_1\Gamma_2\Gamma_5\Gamma_6\Gamma_9 - \ i\,\Gamma_1\Gamma_2\Gamma_7\Gamma_8\Gamma_{10} + \Gamma_1\Gamma_2\Gamma_7\Gamma_8\Gamma_9 \\ & -\Gamma_5\Gamma_6\Gamma_7\Gamma_8\Gamma_9 + \ i\,\Gamma_5\Gamma_6\Gamma_7\Gamma_8\Gamma_{10} + \Gamma_3\Gamma_4\Gamma_7\Gamma_8\Gamma_9 \\ & -i\,\Gamma_3\Gamma_4\Gamma_7\Gamma_8\Gamma_{10} - \Gamma_3\Gamma_4\Gamma_5\Gamma_6\Gamma_9 + i\,\Gamma_3\Gamma_4\Gamma_5\Gamma_6\Gamma_{10}\ ) \end{aligned} \tag{7.14}$$

It is seen from the expressions that $\mathbf{\Gamma}^{15}_{\uparrow\downarrow}$ and $\mathbf{\Gamma}^{15}_{\downarrow\uparrow}$ are not charge conjugated. The below given commutation relations are also useful. We have

$$\begin{array}{llllllll} \left[\mathbf{\Gamma}^{15}_{\uparrow\downarrow}, L_3\right] & = & +\frac{1}{2} & \mathbf{\Gamma}^{15}_{\uparrow\downarrow} & \qquad \left[\mathbf{\Gamma}^{15}_{\downarrow\uparrow}, L_3\right] & = & -\frac{1}{2} & \mathbf{\Gamma}^{15}_{\downarrow\uparrow} \\ \left[\mathbf{\Gamma}^{15}_{\uparrow\downarrow}, R_3\right] & = & -\frac{1}{2} & \mathbf{\Gamma}^{15}_{\uparrow\downarrow} & \qquad \left[\mathbf{\Gamma}^{15}_{\downarrow\uparrow}, R_3\right] & = & +\frac{1}{2} & \mathbf{\Gamma}^{15}_{\downarrow\uparrow} \\ \left[\mathbf{\Gamma}^{15}_{\uparrow\downarrow}, U_3\right] & = & 0 & \mathbf{\Gamma}^{15}_{\uparrow\downarrow} & \qquad \left[\mathbf{\Gamma}^{15}_{\downarrow\uparrow}, U_3\right] & = & 0 & \mathbf{\Gamma}^{15}_{\downarrow\uparrow} \\ \left[\mathbf{\Gamma}^{15}_{\uparrow\downarrow}, U_8\right] & = & 0 & \mathbf{\Gamma}^{15}_{\uparrow\downarrow} & \qquad \left[\mathbf{\Gamma}^{15}_{\downarrow\uparrow}, U_8\right] & = & 0 & \mathbf{\Gamma}^{15}_{\downarrow\uparrow} \\ \left[\mathbf{\Gamma}^{15}_{\uparrow\downarrow}, U_{15}\right] & = & 0 & \mathbf{\Gamma}^{15}_{\uparrow\downarrow} & \qquad \left[\mathbf{\Gamma}^{15}_{\downarrow\uparrow}, U_{15}\right] & = & 0 & \mathbf{\Gamma}^{15}_{\downarrow\uparrow} \end{array} \tag{7.15}$$

If we utilize the Γ_i matrices given by Basis C, then the $\mathbf{\Gamma}^{15}_{\uparrow\downarrow}$ and $\mathbf{\Gamma}^{15}_{\downarrow\uparrow}$ matrices turn out to assume the following form

$$\begin{aligned} &\mathbf{\Gamma}^{15}_{\uparrow\downarrow} = \begin{bmatrix} 0 & A \\ 0 & 0 \end{bmatrix} \quad A = \begin{bmatrix} B & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & B \end{bmatrix} \quad B = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -3 \end{bmatrix} \\ &\mathbf{\Gamma}^{15}_{\downarrow\uparrow} = \begin{bmatrix} 0 & A' \\ 0 & 0 \end{bmatrix} \quad A' = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & B & 0 & 0 \\ 0 & 0 & B & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -3 \end{bmatrix} \end{aligned} \tag{7.16}$$

Here A is a 16×16 block and B is of size 4×4. It turns out that B is proportional to the generator $\mathbf{U}_{15}$ of the fundamental representation of $SU(4)$ which was given in Table (3.2). This can also be seen from the multiplicities appearing in eq. (7.9). From the above matrices it is seen that $\Phi^{15}_{\uparrow\downarrow}$ couples f_L (only left-isospin up states) with f_R (only right-isospin up states) and also $(f^c)_R$ (only left-isospin down states) with $(f^c)_L$ (only right-isospin down states) . Similarly $\Phi^{15}_{\downarrow\uparrow}$ couples couples f_L (only left-isospin down states) with f_R (only right-isospin down states) and also $(f^c)_R$ (only left-isospin up states) with $(f^c)_L$ (only right-isospin up states). Also the hermitian conjugates must be considered. We will come back to this point later in § 7.2. The $\Phi^{15}_{\downarrow\uparrow}$ and $\Phi^{15}_{\uparrow\downarrow}$ Higgs fields can break the following initial symmetries to the final symmetries:

(i) $SU(4) \times SU(2)_L \times SU(2)_R$ down to $SU(3)_c \times U(1)_{L+R} \times U(1)_{B-L}$, where $U(1)_{L+R}$ was previously defined in § 6.

(ii) $SU(3)_c \times SU(2)_L \times SU(2)_R \times U(1)_{B-L}$ down to $SU(3)_c \times U(1)_{L+R} \times U(1)_{B-L}$.

(iii) $SU(3) \times SU(2)_L \times U(1)_Y$ down to $SU(3) \times U(1)_Q$.

(iv) $SU(3)_c \times SU(2)_R \times U(1)_{Y'}$ down to $SU(3) \times U(1)_Q$, where the $U(1)_{Y'}$ was previously defined in §6.

The above chains of symmetry breakdown given in (i) to (iv) are illustrated together with the previous $(2, 2, 1)$ of the 10 and the $(1, 3, 10) \oplus (3, 1, \bar{10})$ of the 126 in Fig. (11.1).

7.2 *Features of the* Φ_{2215}

Some other feature that is worth discussing is how the Φ_{2215} Higgs fields given in eq. (7.12) behave in the Yukawa sector and how they behave in the Higgs mechanism.

(i) Consider the most general case in which we break the $SO(10)$ gauge group with the Higgs field ($\mathbf{\Gamma}^{15}_{\downarrow\uparrow}\ \Phi^{15}_{\downarrow\uparrow} + \mathbf{\Gamma}^{15}_{\uparrow\downarrow}\ \Phi^{15}_{\uparrow\downarrow}$), then the resulting vev of the *first* term endows the following gauge boson of the 45 with mass: W^+_L, W^0_L, W^-_L, W^+_R, W^0_R, W^-_R, A'_r, A'_g, A'_b, Y_r, Y_g,Y_b, Y'_r, Y'_g, Y'_b and X_r, X_g, X_b.
The *second* term endows the following gauge boson of the 45 with mass: W^+_L, W^0_L, W^-_L, W^+_R, W^0_R, W^-_R, A_r, A_g, A_b, A'_r, A'_g, A'_b, Y_r, Y_g, Y_b and X_r, X_g, X_b. The *remark* in § 6.2 applies here as well.

(ii) The first component of the above Higgs field gives rise to mixing among $W^0_R - W^0_L$, where the sign "$-$" is always illustrative. This mixed state has one massive and one massless mode. The second component of the above Higgs field also leads to the mixing among $W^0_R - W^0_L$, and has also one massive and one massless mode.

(iii) If both Higgs field receive vevs then they gives rise to mixing: $W^0_R - W^0_L$ where this state has one massive and one massless mode. $W^\pm_R - W^\pm_L$ with two massive modes for each charge configuration. $A'_\alpha - Y_\alpha$ with two massive modes for each color configuration. The two massive modes are understood to be high and low masses.

(iv) As pointed out previously, the first Higgs field in ($\mathbf{\Gamma}^{15}_{\downarrow\uparrow}\ \Phi^{15}_{\downarrow\uparrow} + \mathbf{\Gamma}^{15}_{\uparrow\downarrow}\ \Phi^{15}_{\uparrow\downarrow}$) couples only to the (d, e) (down-type) fermions and the second term couples only to (u, ν) (up-type) fermions. Thereby all the fermions are endowed with Dirac masses. Let us consider the Yukawa term $Y_{ij}\ \left(\bar{\Psi}_i\ \Phi_{2215}\ \Psi_j\right)$ where $i, j = (1, 2, 3)$ are family indices, Y_{ij} are the Yukawa couplings and Ψ_i are the family spinors. The vacuum expectation values of the $\mathbf{\Gamma}^{15}_{\downarrow\uparrow}\ \Phi^{15}_{\downarrow\uparrow} + \mathbf{\Gamma}^{15}_{\uparrow\downarrow}\ \Phi^{15}_{\uparrow\downarrow}$ Higgs fields endow all the fermions with Dirac masses. The vertices generated by the Yukawa term are shown in Fig. (7.2). The charges conserved at each vertex can be checked from

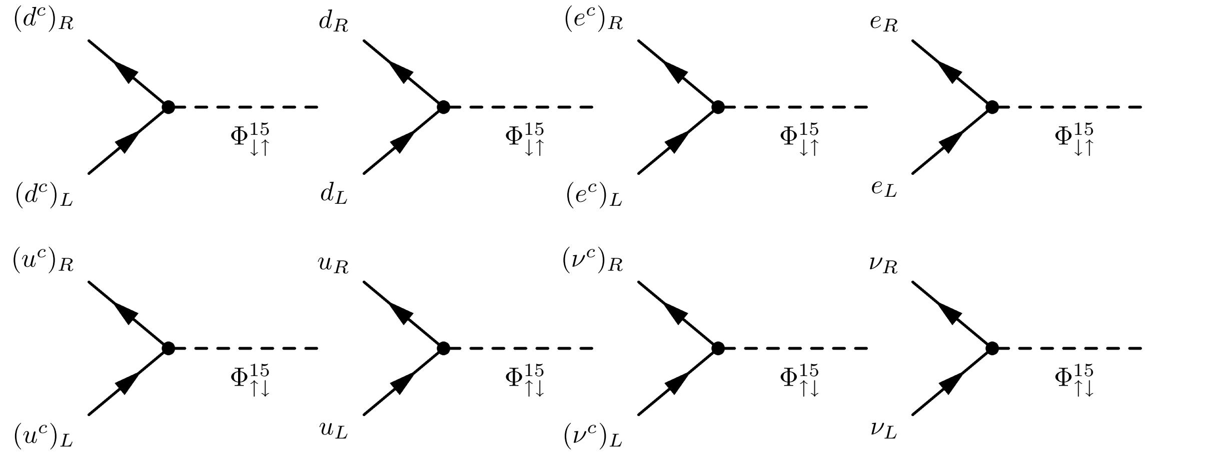

Fig. 7.2: The Scalars $\Phi^{15}_{\downarrow\uparrow}$ and $\Phi^{15}_{\uparrow\downarrow}$ couple to d, e and u, ν fermions respectively. All scalars flow out of the vertices.

Tables, (4.3), (4.4)and (7.1). The $(2, 2, 15)$ multiplet has a multiplicity of 3 at the positions that couple to leptons, therefore the relations $m_d = 3\,m_e$ and $m_u = 3\,m_\nu$ will hold. Since $\Phi^{15}_{\downarrow\uparrow}$ produces only Dirac mass terms for (d, e) (down-type) and $\Phi^{15}_{\uparrow\downarrow}$ produces only Dirac mass terms for (u, ν) (up-type) fermions. It follows that the down fermion sector and the up fermion sector can get different masses i.e., m_d and m_u can be different, provided that we employ not only both Higgs fields but assign them unequal vevs. One should have in mind that $\Phi^{15}_{\downarrow\uparrow}$ and $\Phi^{15}_{\uparrow\downarrow}$ are not charge conjugated and their vevs can assume different values. This feature differentiates the $(2, 2, 15)$ singlet of the 126 from the $(2, 2, 1)$of the 10 where the Φ_5 and $\bar{\Phi}_5$ are conjugated.

Details concerning the above features will be visited later on again in § 11 and § 12.

7.3 Weight Diagrams for the 126

Charges of some 126 Higgs Fields											
$(3, 1, \overline{10}) \oplus (1, 3, 10)$	Q	B-L	I_{3R}	I_{3L}	Y	(2,2,15)	Q	B-L	I_{3R}	I_{3L}	Y
$\Phi^{10}_{\downarrow 0}$	0	+2	0	-1	+2	$\Phi^{15}_{\uparrow\uparrow}$	+1	0	+1/2	+1/2	+1
$\Phi^{10}_{\hat{0} 0}$	+1	+2	0	0	+2	$\Phi^{15}_{\downarrow\uparrow}$	0	0	+1/2	-1/2	+1
$\Phi^{10}_{\uparrow 0}$	+2	+2	0	+1	+2	$\Phi^{15}_{\uparrow\downarrow}$	0	0	-1/2	+1/2	-1
$\Phi^{10}_{0\downarrow}$	-2	-2	-1	0	-4	$\Phi^{15}_{\downarrow\downarrow}$	-1	0	-1/2	-1/2	-1
$\Phi^{10}_{0\hat{0}}$	-1	-2	0	0	-2						
$\Phi^{10}_{0\uparrow}$	0	-2	+1	0	0						

Tab. 7.1: Charges of some Higgs bosons in the 126 Higgs representation of $SO(10)$

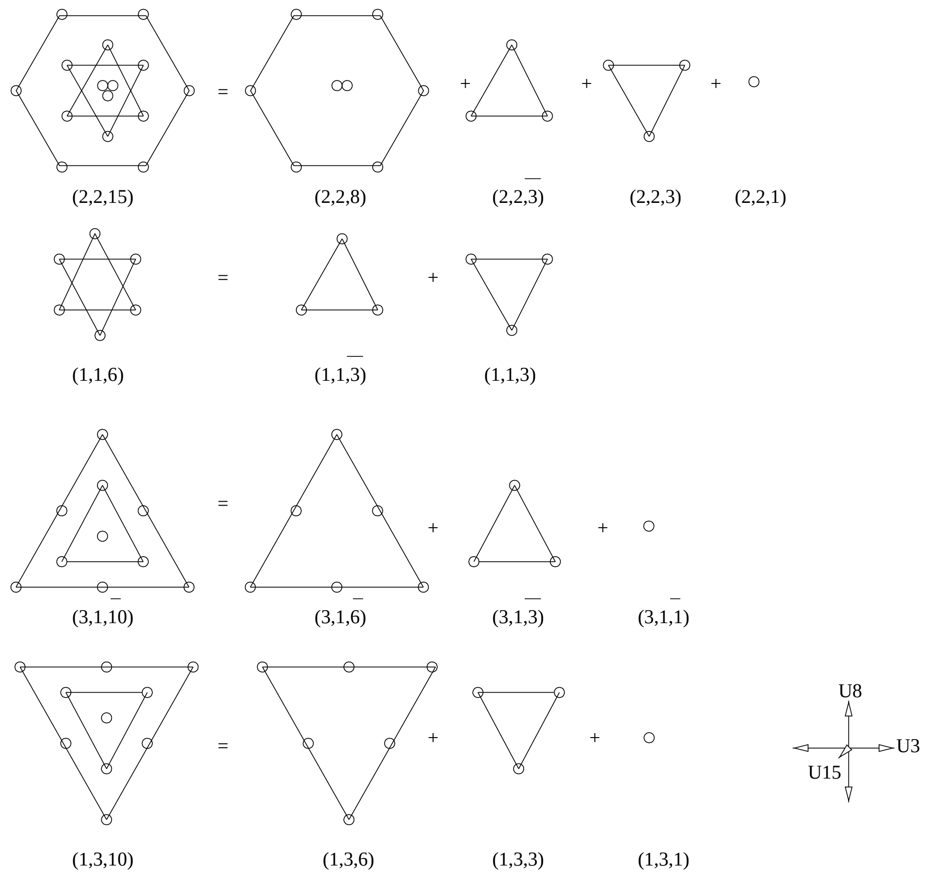

Fig. 7.3: In the figure L, R isospin weights are suppressed. U_{15} points out of page. U_8 and U_3 are laying on the page. The decomposition of the 126 with respect to $SU(2)_L \times SU(2)_R \times SU(4)_c$ is given in the first column. The decomposition of the 126 with respect to $SU(2)_L \times SU(2)_R \times SU(3)_c \times U(1)_{B-L}$ are given horizontally.

8. THE HIGGS MULTIPLETS: THE (210)-REPRESENTATION

In this part we elaborate the 210 Higgs representation. The 210 is spanned by the 4-products of the Γ_i basis of $SO(10)$ which was introduced in § 2.3. These 4-products include all the possible $\Gamma_i\Gamma_j\Gamma_k\Gamma_l$ terms in which $i \neq j \neq k \neq l$. There will be 210 such components. We have

$$\frac{1}{\sqrt{32}}\,\phi_{ijkl}\;\Gamma_i\Gamma_j\Gamma_k\Gamma_l = \frac{1}{16}\left(\mathbf{\Gamma}\cdot\Phi + \mathbf{\Gamma}^{\dagger}\cdot\Phi^{\dagger}\right) = \begin{pmatrix} \Omega_{11} & \Omega_{12} \\ \Omega_{21} & \Omega_{22} \end{pmatrix} \tag{8.1}$$

The 210 Higgs representation is indeed rather complex and contains only a few useful multiplets for the $SO(10)$ Higgs sector. These fields decompose under the $SU(2)_L\times SU(2)_R\times SU(4)$ gauge group into $(2,2,10)$ + $(2,2,\bar{10})$ + $(1,3,\bar{15})$ + $(3,1,15)$ + $(1,1,15)$ + $(2,2,6)$ + $(1,1,1)$ multiplets [53]. Many of these fields occupy the same sites, if one considers an explicit matrix representation Σ. The Ω_{11} block is presented in two parts below such that the former one contains the Higgs fields of the first five multiplets i.e., $(2,2,10)$ + $(2,2,\bar{10})$+$(1,3,\bar{15})$ + $(3,1,15)$ +$(1,1,1)$ and the latter contains the remaining 2 multiplets i.e., $(1,1,15)$ + $(2,2,6)$ where we have used basis C. The conventions used for the upper and lower scripts in the Higgs fields was previously defined in § 7 and will not be repeated here. The sum of the two gives Ω_{11}. For basis C, Ω_{12} and Ω_{21} are null blocks. Ω_{22} is related to Ω_{11} through a parity transformation where P is given in eq. (4.25). The diagonal elements λ_i for $i=(1,\ldots,16)$ appearing in the first matrix below are explicitly

$$\begin{aligned}
\lambda_1 &= -\Phi^s_{00} + \Phi^7_{\hat{0}0} + \Phi^{15}_{\hat{0}0} & \lambda_9 &= +\Phi^s_{00} + \Phi^7_{0\hat{0}} + \Phi^{15}_{0\hat{0}} \\
\lambda_2 &= -\Phi^s_{00} + \Phi^8_{\hat{0}0} + \Phi^{15}_{\hat{0}0} & \lambda_{10} &= +\Phi^s_{00} + \Phi^8_{0\hat{0}} + \Phi^{15}_{0\hat{0}} \\
\lambda_3 &= -\Phi^s_{00} - \Phi^7_{\hat{0}0} - \Phi^8_{\hat{0}0} + \Phi^{15}_{\hat{0}0} & \lambda_{11} &= +\Phi^s_{00} - \Phi^7_{0\hat{0}} - \Phi^8_{0\hat{0}} + \Phi^{15}_{0\hat{0}} \\
\lambda_4 &= -\Phi^s_{00} - 3\,\Phi^{15}_{\hat{0}0} & \lambda_{12} &= +\Phi^s_{00} - 3\,\Phi^{15}_{0\hat{0}} \\
\lambda_5 &= -\Phi^s_{00} - \Phi^7_{\hat{0}0} - \Phi^{15}_{\hat{0}0} & \lambda_{13} &= +\Phi^s_{00} - \Phi^7_{0\hat{0}} - \Phi^{15}_{0\hat{0}} \\
\lambda_6 &= -\Phi^s_{00} - \Phi^8_{\hat{0}0} - \Phi^{15}_{\hat{0}0} & \lambda_{14} &= +\Phi^s_{00} - \Phi^8_{0\hat{0}} - \Phi^{15}_{0\hat{0}} \\
\lambda_7 &= -\Phi^s_{00} + \Phi^7_{\hat{0}0} + \Phi^8_{\hat{0}0} - \Phi^{15}_{\hat{0}0} & \lambda_{15} &= +\Phi^s_{00} + \Phi^7_{0\hat{0}} + \Phi^8_{0\hat{0}} - \Phi^{15}_{0\hat{0}} \\
\lambda_8 &= -\Phi^s_{00} + 3\,\Phi^{15}_{\hat{0}0} & \lambda_{16} &= +\Phi^s_{00} + 3\,\Phi^{15}_{0\hat{0}}
\end{aligned} \tag{8.2}$$

The diagonal elements α_i and β_i for $i=(1,\ldots,8)$ in the same matrix are explicitly

$$\begin{aligned}
\beta_1 &= +\Phi^7_{\uparrow 0} - \Phi^{15}_{\uparrow 0} & \beta_5 &= +\Phi^7_{\downarrow 0} - \Phi^{15}_{\downarrow 0} \\
\beta_2 &= +\Phi^8_{\uparrow 0} - \Phi^{15}_{\uparrow 0} & \beta_6 &= +\Phi^8_{\downarrow 0} - \Phi^{15}_{\downarrow 0} \\
\beta_3 &= -\Phi^7_{\uparrow 0} - \Phi^8_{\uparrow 0} - \Phi^{15}_{\uparrow 0} & \beta_7 &= -\Phi^7_{\downarrow 0} - \Phi^8_{\downarrow 0} - \Phi^{15}_{\downarrow 0} \\
\beta_4 &= +3\Phi^{15}_{\uparrow 0} & \beta_8 &= +3\Phi^{15}_{\downarrow 0}
\end{aligned} \tag{8.3}$$

$$\begin{aligned}
\alpha_1 &= +\Phi^7_{0\uparrow} - \Phi^{15}_{0\uparrow} & \alpha_5 &= +\Phi^7_{0\downarrow} - \Phi^{15}_{0\downarrow} \\
\alpha_2 &= +\Phi^8_{0\uparrow} - \Phi^{15}_{0\uparrow} & \alpha_6 &= +\Phi^8_{0\downarrow} - \Phi^{15}_{0\downarrow} \\
\alpha_3 &= -\Phi^7_{0\uparrow} - \Phi^8_{0\uparrow} - \Phi^{15}_{0\uparrow} & \alpha_7 &= -\Phi^7_{0\downarrow} - \Phi^8_{0\downarrow} - \Phi^{15}_{0\downarrow 0} \\
\alpha_4 &= +3\Phi^{15}_{\uparrow 0} & \alpha_8 &= +3\Phi^{15}_{0\downarrow}
\end{aligned} \tag{8.4}$$

Finally the diagonal elements Λ_i for $i=(1,\ldots,4)$ in the second matrix are

$$\begin{aligned}
\Lambda_1 &= +\Phi^7_{00} + \Phi^{15}_{00} \\
\Lambda_2 &= +\Phi^8_{00} + \Phi^{15}_{00} \\
\Lambda_3 &= -\Phi^7_{00} - \Phi^8_{00} + \Phi^{15}_{00} \\
\Lambda_4 &= -3\Phi^{15}_{00}
\end{aligned} \tag{8.5}$$

$$
\begin{bmatrix}
\lambda_1 & \Phi^{4}_{\hat{0}0} & \Phi^{5}_{\hat{0}0} & \Phi^{12}_{\hat{0}0} & \beta_5 & \Phi^{4}_{\downarrow 0} & \Phi^{5}_{\downarrow 0} & \Phi^{12}_{\downarrow 0} & \Phi^{7}_{\downarrow\uparrow} & \Phi^{1}_{\downarrow\uparrow} & \Phi^{2}_{\downarrow\uparrow} & \Phi^{4}_{\downarrow\uparrow} & \Phi^{7}_{\downarrow\downarrow} & \Phi^{1}_{\downarrow\downarrow} & \Phi^{2}_{\downarrow\downarrow} & \Phi^{4}_{\downarrow\downarrow} \\
\Phi^{1}_{\hat{0}0} & \lambda_2 & \Phi^{6}_{\hat{0}0} & \Phi^{13}_{\hat{0}0} & \Phi^{1}_{\downarrow 0} & \beta_6 & \Phi^{6}_{\downarrow 0} & \Phi^{13}_{\downarrow 0} & \Phi^{1}_{\downarrow\uparrow} & \Phi^{8}_{\downarrow\uparrow} & \Phi^{3}_{\downarrow\uparrow} & \Phi^{5}_{\downarrow\uparrow} & \Phi^{1}_{\downarrow\downarrow} & \Phi^{8}_{\downarrow\downarrow} & \Phi^{3}_{\downarrow\downarrow} & \Phi^{5}_{\downarrow\downarrow} \\
\Phi^{2}_{\hat{0}0} & \Phi^{3}_{\hat{0}0} & \lambda_3 & \Phi^{14}_{\hat{0}0} & \Phi^{2}_{\downarrow 0} & \Phi^{3}_{\downarrow 0} & \beta_7 & \Phi^{14}_{\downarrow 0} & \Phi^{2}_{\downarrow\uparrow} & \Phi^{3}_{\downarrow\uparrow} & \Phi^{9}_{\downarrow\uparrow} & \Phi^{6}_{\downarrow\uparrow} & \Phi^{2}_{\downarrow\downarrow} & \Phi^{3}_{\downarrow\downarrow} & \Phi^{9}_{\downarrow\downarrow} & \Phi^{6}_{\downarrow\downarrow} \\
\Phi^{9}_{\hat{0}0} & \Phi^{10}_{\hat{0}0} & \Phi^{11}_{\hat{0}0} & \lambda_4 & \Phi^{9}_{\downarrow 0} & \Phi^{10}_{\downarrow 0} & \Phi^{11}_{\downarrow 0} & \beta_8 & \Phi^{4}_{\downarrow\uparrow} & \Phi^{5}_{\downarrow\uparrow} & \Phi^{6}_{\downarrow\uparrow} & \Phi^{10}_{\downarrow\uparrow} & \Phi^{4}_{\downarrow\downarrow} & \Phi^{5}_{\downarrow\downarrow} & \Phi^{6}_{\downarrow\downarrow} & \Phi^{10}_{\downarrow\downarrow} \\
\beta_1 & \Phi^{4}_{\uparrow 0} & \Phi^{5}_{\uparrow 0} & \Phi^{12}_{\uparrow 0} & \lambda_5 & -\Phi^{4}_{\hat{0}0} & -\Phi^{5}_{\hat{0}0} & -\Phi^{12}_{\hat{0}0} & \Phi^{7}_{\uparrow\uparrow} & \Phi^{1}_{\uparrow\uparrow} & \Phi^{2}_{\uparrow\uparrow} & \Phi^{4}_{\uparrow\uparrow} & \Phi^{7}_{\uparrow\downarrow} & \Phi^{1}_{\uparrow\downarrow} & \Phi^{2}_{\uparrow\downarrow} & \Phi^{4}_{\uparrow\downarrow} \\
\Phi^{1}_{\uparrow 0} & \beta_2 & \Phi^{6}_{\uparrow 0} & \Phi^{13}_{\uparrow 0} & -\Phi^{1}_{\hat{0}0} & \lambda_6 & -\Phi^{6}_{\hat{0}0} & -\Phi^{13}_{\hat{0}0} & \Phi^{1}_{\uparrow\uparrow} & \Phi^{8}_{\uparrow\uparrow} & \Phi^{3}_{\uparrow\uparrow} & \Phi^{5}_{\uparrow\uparrow} & \Phi^{1}_{\uparrow\downarrow} & \Phi^{8}_{\uparrow\downarrow} & \Phi^{3}_{\uparrow\downarrow} & \Phi^{5}_{\uparrow\downarrow} \\
\Phi^{2}_{\uparrow 0} & \Phi^{3}_{\uparrow 0} & \beta_3 & \Phi^{14}_{\uparrow 0} & -\Phi^{2}_{\hat{0}0} & -\Phi^{3}_{\hat{0}0} & \lambda_7 & -\Phi^{14}_{\hat{0}0} & \Phi^{2}_{\uparrow\uparrow} & \Phi^{3}_{\uparrow\uparrow} & \Phi^{9}_{\uparrow\uparrow} & \Phi^{6}_{\uparrow\uparrow} & \Phi^{2}_{\uparrow\downarrow} & \Phi^{3}_{\uparrow\downarrow} & \Phi^{9}_{\uparrow\downarrow} & \Phi^{6}_{\uparrow\downarrow} \\
\Phi^{9}_{\uparrow 0} & \Phi^{10}_{\uparrow 0} & \Phi^{11}_{\uparrow 0} & \beta_4 & -\Phi^{9}_{\hat{0}0} & -\Phi^{10}_{\hat{0}0} & -\Phi^{11}_{\hat{0}0} & \lambda_8 & \Phi^{4}_{\uparrow\uparrow} & \Phi^{5}_{\uparrow\uparrow} & \Phi^{6}_{\uparrow\uparrow} & \Phi^{10}_{\uparrow\uparrow} & \Phi^{4}_{\uparrow\downarrow} & \Phi^{5}_{\uparrow\downarrow} & \Phi^{6}_{\uparrow\downarrow} & \Phi^{10}_{\uparrow\downarrow} \\
\Phi^{7}_{\uparrow\downarrow} & \Phi^{1}_{\uparrow\downarrow} & \Phi^{2}_{\uparrow\downarrow} & \Phi^{4}_{\uparrow\downarrow} & \Phi^{7}_{\downarrow\downarrow} & \Phi^{1}_{\downarrow\downarrow} & \Phi^{2}_{\downarrow\downarrow} & \Phi^{4}_{\downarrow\downarrow} & \lambda_9 & \Phi^{4}_{0\hat{0}} & \Phi^{5}_{0\hat{0}} & \Phi^{12}_{0\hat{0}} & \alpha_5 & \Phi^{4}_{0\downarrow} & \Phi^{5}_{0\downarrow} & \Phi^{12}_{0\downarrow} \\
\Phi^{1}_{\uparrow\downarrow} & \Phi^{8}_{\uparrow\downarrow} & \Phi^{3}_{\uparrow\downarrow} & \Phi^{5}_{\uparrow\downarrow} & \Phi^{1}_{\downarrow\downarrow} & \Phi^{8}_{\downarrow\downarrow} & \Phi^{3}_{\downarrow\downarrow} & \Phi^{5}_{\downarrow\downarrow} & \Phi^{1}_{0\hat{0}} & \lambda_{10} & \Phi^{6}_{0\hat{0}} & \Phi^{13}_{0\hat{0}} & \Phi^{1}_{0\downarrow} & \alpha_6 & \Phi^{6}_{0\downarrow} & \Phi^{13}_{0\downarrow} \\
\Phi^{2}_{\uparrow\downarrow} & \Phi^{3}_{\uparrow\downarrow} & \Phi^{9}_{\uparrow\downarrow} & \Phi^{6}_{\uparrow\downarrow} & \Phi^{2}_{\downarrow\downarrow} & \Phi^{3}_{\downarrow\downarrow} & \Phi^{9}_{\downarrow\downarrow} & \Phi^{6}_{\downarrow\downarrow} & \Phi^{2}_{0\hat{0}} & \Phi^{3}_{0\hat{0}} & \lambda_{11} & \Phi^{14}_{0\hat{0}} & \Phi^{2}_{0\downarrow} & \Phi^{3}_{0\downarrow} & \alpha_7 & \Phi^{12}_{0\downarrow} \\
\Phi^{4}_{\uparrow\downarrow} & \Phi^{5}_{\uparrow\downarrow} & \Phi^{6}_{\uparrow\downarrow} & \Phi^{10}_{\uparrow\downarrow} & \Phi^{4}_{\downarrow\downarrow} & \Phi^{5}_{\downarrow\downarrow} & \Phi^{6}_{\downarrow\downarrow} & \Phi^{10}_{\downarrow\downarrow} & \Phi^{9}_{0\hat{0}} & \Phi^{10}_{0\hat{0}} & \Phi^{11}_{0\hat{0}} & \lambda_{12} & \Phi^{9}_{0\downarrow} & \Phi^{10}_{0\downarrow} & \Phi^{11}_{0\downarrow} & \alpha_8 \\
\Phi^{7}_{\uparrow\uparrow} & \Phi^{1}_{\uparrow\uparrow} & \Phi^{2}_{\uparrow\uparrow} & \Phi^{4}_{\uparrow\uparrow} & \Phi^{7}_{\downarrow\uparrow} & \Phi^{1}_{\downarrow\uparrow} & \Phi^{2}_{\downarrow\uparrow} & \Phi^{4}_{\downarrow\uparrow} & \alpha_1 & \Phi^{4}_{0\uparrow} & \Phi^{5}_{0\uparrow} & \Phi^{12}_{0\uparrow} & \lambda_{13} & -\Phi^{4}_{0\hat{0}} & -\Phi^{5}_{0\hat{0}} & -\Phi^{12}_{0\hat{0}} \\
\Phi^{1}_{\uparrow\uparrow} & \Phi^{8}_{\uparrow\uparrow} & \Phi^{3}_{\uparrow\uparrow} & \Phi^{5}_{\uparrow\uparrow} & \Phi^{1}_{\downarrow\uparrow} & \Phi^{8}_{\downarrow\uparrow} & \Phi^{3}_{\downarrow\uparrow} & \Phi^{5}_{\downarrow\uparrow} & \Phi^{1}_{0\uparrow} & \alpha_2 & \Phi^{6}_{0\uparrow} & \Phi^{13}_{0\uparrow} & -\Phi^{1}_{0\hat{0}} & \lambda_{14} & -\Phi^{6}_{0\hat{0}} & -\Phi^{13}_{0\hat{0}} \\
\Phi^{2}_{\uparrow\uparrow} & \Phi^{3}_{\uparrow\uparrow} & \Phi^{9}_{\uparrow\uparrow} & \Phi^{6}_{\uparrow\uparrow} & \Phi^{2}_{\downarrow\uparrow} & \Phi^{3}_{\downarrow\uparrow} & \Phi^{9}_{\downarrow\uparrow} & \Phi^{6}_{\downarrow\uparrow} & \Phi^{2}_{0\uparrow} & \Phi^{3}_{0\uparrow} & \alpha_3 & \Phi^{14}_{0\uparrow} & -\Phi^{2}_{0\hat{0}} & -\Phi^{3}_{0\hat{0}} & \lambda_{15} & -\Phi^{14}_{0\hat{0}} \\
\Phi^{4}_{\uparrow\uparrow} & \Phi^{5}_{\uparrow\uparrow} & \Phi^{6}_{\uparrow\uparrow} & \Phi^{10}_{\uparrow\uparrow} & \Phi^{4}_{\downarrow\uparrow} & \Phi^{5}_{\downarrow\uparrow} & \Phi^{6}_{\downarrow\uparrow} & \Phi^{10}_{\downarrow\uparrow} & \Phi^{9}_{0\uparrow} & \Phi^{10}_{0\uparrow} & \Phi^{11}_{0\uparrow} & \alpha_4 & -\Phi^{9}_{0\hat{0}} & -\Phi^{10}_{0\hat{0}} & -\Phi^{11}_{0\hat{0}} & \lambda_{16}
\end{bmatrix}
$$

$$
\begin{bmatrix}
\Lambda_1 & \Phi^{1}_{00} & \Phi^{2}_{00} & \Phi^{12}_{00} & 0 & 0 & 0 & 0 & 0 & \Phi^{1}_{\downarrow\uparrow} & \Phi^{2}_{\downarrow\uparrow} & \Phi^{3}_{\downarrow\uparrow} & 0 & \Phi^{1}_{\downarrow\downarrow} & \Phi^{2}_{\downarrow\downarrow} & \Phi^{3}_{\downarrow\downarrow} \\
\Phi^{4}_{00} & \Lambda_2 & \Phi^{3}_{00} & \Phi^{13}_{00} & 0 & 0 & 0 & 0 & \Phi^{1}_{\downarrow\uparrow} & 0 & \Phi^{6}_{\downarrow\uparrow} & \Phi^{5}_{\downarrow\uparrow} & \Phi^{1}_{\downarrow\downarrow} & 0 & \Phi^{6}_{\downarrow\downarrow} & \Phi^{5}_{\downarrow\downarrow} \\
\Phi^{5}_{00} & \Phi^{6}_{00} & \Lambda_3 & \Phi^{14}_{00} & 0 & 0 & 0 & 0 & \Phi^{2}_{\downarrow\uparrow} & \Phi^{6}_{\downarrow\uparrow} & 0 & \Phi^{4}_{\downarrow\uparrow} & \Phi^{2}_{\downarrow\downarrow} & \Phi^{6}_{\downarrow\downarrow} & 0 & \Phi^{4}_{\downarrow\downarrow} \\
\Phi^{9}_{00} & \Phi^{10}_{00} & \Phi^{11}_{00} & \Lambda_4 & 0 & 0 & 0 & 0 & \Phi^{3}_{\downarrow\uparrow} & \Phi^{5}_{\downarrow\uparrow} & \Phi^{4}_{\downarrow\uparrow} & 0 & \Phi^{3}_{\downarrow\downarrow} & \Phi^{5}_{\downarrow\downarrow} & \Phi^{4}_{\downarrow\downarrow} & 0 \\
0 & 0 & 0 & 0 & \Lambda_1 & -\Phi^{1}_{00} & -\Phi^{2}_{00} & -\Phi^{12}_{00} & 0 & \Phi^{1}_{\uparrow\uparrow} & \Phi^{2}_{\uparrow\uparrow} & \Phi^{3}_{\uparrow\uparrow} & 0 & \Phi^{1}_{\uparrow\downarrow} & \Phi^{2}_{\uparrow\downarrow} & \Phi^{3}_{\uparrow\downarrow} \\
0 & 0 & 0 & 0 & -\Phi^{4}_{00} & \Lambda_2 & -\Phi^{3}_{00} & -\Phi^{13}_{00} & \Phi^{1}_{\uparrow\uparrow} & 0 & \Phi^{6}_{\uparrow\uparrow} & \Phi^{5}_{\uparrow\uparrow} & \Phi^{1}_{\uparrow\downarrow} & 0 & \Phi^{6}_{\uparrow\downarrow} & \Phi^{5}_{\uparrow\downarrow} \\
0 & 0 & 0 & 0 & -\Phi^{5}_{00} & -\Phi^{6}_{00} & \Lambda_3 & -\Phi^{14}_{00} & \Phi^{2}_{\uparrow\uparrow} & \Phi^{6}_{\uparrow\uparrow} & 0 & \Phi^{4}_{\uparrow\uparrow} & \Phi^{2}_{\uparrow\downarrow} & \Phi^{6}_{\uparrow\downarrow} & 0 & \Phi^{4}_{\uparrow\downarrow} \\
0 & 0 & 0 & 0 & -\Phi^{9}_{00} & -\Phi^{10}_{00} & -\Phi^{11}_{00} & \Lambda_4 & \Phi^{3}_{\uparrow\uparrow} & \Phi^{5}_{\uparrow\uparrow} & \Phi^{4}_{\uparrow\uparrow} & 0 & \Phi^{3}_{\uparrow\downarrow} & \Phi^{5}_{\uparrow\downarrow} & \Phi^{4}_{\uparrow\downarrow} & 0 \\
0 & \Phi^{4}_{\uparrow\downarrow} & \Phi^{5}_{\uparrow\downarrow} & \Phi^{6}_{\uparrow\downarrow} & 0 & \Phi^{4}_{\downarrow\downarrow} & \Phi^{5}_{\downarrow\downarrow} & \Phi^{6}_{\downarrow\downarrow} & \Lambda_1 & \Phi^{4}_{00} & \Phi^{5}_{00} & \Phi^{9}_{00} & 0 & 0 & 0 & 0 \\
\Phi^{4}_{\uparrow\downarrow} & 0 & {\Phi_{\uparrow\downarrow}}^{3} & \Phi^{2}_{\uparrow\downarrow} & \Phi^{4}_{\downarrow\downarrow} & 0 & \Phi^{3}_{\downarrow\downarrow} & \Phi^{2}_{\downarrow\downarrow} & \Phi^{1}_{00} & \Lambda_2 & \Phi^{6}_{00} & \Phi^{10}_{00} & 0 & 0 & 0 & 0 \\
\Phi^{5}_{\uparrow\downarrow} & \Phi^{3}_{\uparrow\downarrow} & 0 & \Phi^{1}_{\uparrow\downarrow} & \Phi^{5}_{\downarrow\downarrow} & \Phi^{3}_{\downarrow\downarrow} & 0 & \Phi^{1}_{\downarrow\downarrow} & \Phi^{2}_{00} & \Phi^{3}_{00} & \Lambda_3 & \Phi^{11}_{00} & 0 & 0 & 0 & 0 \\
\Phi^{6}_{\uparrow\downarrow} & \Phi^{2}_{\uparrow\downarrow} & \Phi^{1}_{\uparrow\downarrow} & 0 & \Phi^{6}_{\downarrow\downarrow} & \Phi^{2}_{\downarrow\downarrow} & \Phi^{1}_{\downarrow\downarrow} & 0 & \Phi^{12}_{00} & \Phi^{13}_{00} & \Phi^{14}_{00} & \Lambda_4 & 0 & 0 & 0 & 0 \\
0 & \Phi^{4}_{\uparrow\uparrow} & \Phi^{5}_{\uparrow\uparrow} & \Phi^{6}_{\uparrow\uparrow} & 0 & \Phi^{4}_{\downarrow\uparrow} & \Phi^{5}_{\downarrow\uparrow} & \Phi^{6}_{\downarrow\uparrow} & 0 & 0 & 0 & 0 & \Lambda_1 & -\Phi^{4}_{00} & -\Phi^{5}_{00} & -\Phi^{9}_{00} \\
\Phi^{4}_{\uparrow\uparrow} & 0 & \Phi^{3}_{\uparrow\uparrow} & \Phi^{2}_{\uparrow\uparrow} & \Phi^{4}_{\downarrow\uparrow} & 0 & \Phi^{3}_{\downarrow\uparrow} & \Phi^{2}_{\downarrow\uparrow} & 0 & 0 & 0 & 0 & -\Phi^{1}_{00} & \Lambda_2 & -\Phi^{6}_{00} & -\Phi^{10}_{00} \\
\Phi^{5}_{\uparrow\uparrow} & \Phi^{3}_{\uparrow\uparrow} & 0 & \Phi^{1}_{\uparrow\uparrow} & \Phi^{5}_{\downarrow\uparrow} & \Phi^{3}_{\downarrow\uparrow} & 0 & \Phi^{1}_{\downarrow\uparrow} & 0 & 0 & 0 & 0 & -\Phi^{2}_{00} & -\Phi^{3}_{00} & \Lambda_3 & -\Phi^{11}_{00} \\
\Phi^{6}_{\uparrow\uparrow} & \Phi^{2}_{\uparrow\uparrow} & \Phi^{1}_{\uparrow\uparrow} & 0 & \Phi^{6}_{\downarrow\uparrow} & \Phi^{2}_{\downarrow\uparrow} & \Phi^{1}_{\downarrow\uparrow} & 0 & 0 & 0 & 0 & 0 & -\Phi^{12}_{00} & -\Phi^{13}_{00} & -\Phi^{14}_{00} & \Lambda_4
\end{bmatrix}
$$

The $(2,2,\bar{10})$ and the $(2,2,10)$ multiplets of the 210 Higgs representation are

$$
\begin{array}{llll}
\Phi^{1}_{(\pm\frac{1}{2},\pm\frac{1}{2},\,0,-\frac{1}{\sqrt{3}},-\frac{1}{\sqrt{6}})} & \Phi^{1}_{(\pm\frac{1}{2},\mp\frac{1}{2},\,0,-\frac{1}{\sqrt{3}},-\frac{1}{\sqrt{6}})} & \Phi^{1}_{(\pm\frac{1}{2},\pm\frac{1}{2},\,0,+\frac{1}{\sqrt{3}},+\frac{1}{\sqrt{6}})} & \Phi^{1}_{(\pm\frac{1}{2},\mp\frac{1}{2},\,0,+\frac{1}{\sqrt{3}},+\frac{1}{\sqrt{6}})} \\
\Phi^{2}_{(\pm\frac{1}{2},\pm\frac{1}{2},-\frac{1}{2},+\frac{1}{2\sqrt{3}},-\frac{1}{\sqrt{6}})} & \Phi^{2}_{(\pm\frac{1}{2},\mp\frac{1}{2},-\frac{1}{2},+\frac{1}{2\sqrt{3}},-\frac{1}{\sqrt{6}})} & \Phi^{2}_{(\pm\frac{1}{2},\pm\frac{1}{2},+\frac{1}{2},-\frac{1}{2\sqrt{3}},+\frac{1}{\sqrt{6}})} & \Phi^{2}_{(\pm\frac{1}{2},\mp\frac{1}{2},+\frac{1}{2},-\frac{1}{2\sqrt{3}},+\frac{1}{\sqrt{6}})} \\
\Phi^{3}_{(\pm\frac{1}{2},\pm\frac{1}{2},-\frac{1}{2},-\frac{1}{2\sqrt{3}},+\frac{1}{\sqrt{6}})} & \Phi^{3}_{(\pm\frac{1}{2},\mp\frac{1}{2},-\frac{1}{2},-\frac{1}{2\sqrt{3}},+\frac{1}{\sqrt{6}})} & \Phi^{3}_{(\pm\frac{1}{2},\pm\frac{1}{2},+\frac{1}{2},+\frac{1}{2\sqrt{3}},-\frac{1}{\sqrt{6}})} & \Phi^{3}_{(\pm\frac{1}{2},\mp\frac{1}{2},+\frac{1}{2},+\frac{1}{2\sqrt{3}},-\frac{1}{\sqrt{6}})} \\
\Phi^{4}_{(\pm\frac{1}{2},\pm\frac{1}{2},\,0,+\frac{1}{\sqrt{3}},+\frac{1}{\sqrt{6}})} & \Phi^{4}_{(\pm\frac{1}{2},\mp\frac{1}{2},\,0,+\frac{1}{\sqrt{3}},+\frac{1}{\sqrt{6}})} & \Phi^{4}_{(\pm\frac{1}{2},\pm\frac{1}{2},\,0,-\frac{1}{\sqrt{3}},-\frac{1}{\sqrt{6}})} & \Phi^{4}_{(\pm\frac{1}{2},\mp\frac{1}{2},\,0,-\frac{1}{\sqrt{3}},-\frac{1}{\sqrt{6}})} \\
\Phi^{5}_{(\pm\frac{1}{2},\pm\frac{1}{2},+\frac{1}{2},-\frac{1}{2\sqrt{3}},+\frac{1}{\sqrt{6}})} & \Phi^{5}_{(\pm\frac{1}{2},\mp\frac{1}{2},+\frac{1}{2},-\frac{1}{2\sqrt{3}},+\frac{1}{\sqrt{6}})} & \Phi^{5}_{(\pm\frac{1}{2},\pm\frac{1}{2},-\frac{1}{2},+\frac{1}{2\sqrt{3}},-\frac{1}{\sqrt{6}})} & \Phi^{5}_{(\pm\frac{1}{2},\mp\frac{1}{2},-\frac{1}{2},+\frac{1}{2\sqrt{3}},-\frac{1}{\sqrt{6}})} \\
\Phi^{6}_{(\pm\frac{1}{2},\pm\frac{1}{2},+\frac{1}{2},+\frac{1}{2\sqrt{3}},-\frac{1}{\sqrt{6}})} & \Phi^{6}_{(\pm\frac{1}{2},\mp\frac{1}{2},+\frac{1}{2},+\frac{1}{2\sqrt{3}},-\frac{1}{\sqrt{6}})} & \Phi^{6}_{(\pm\frac{1}{2},\pm\frac{1}{2},-\frac{1}{2},-\frac{1}{2\sqrt{3}},+\frac{1}{\sqrt{6}})} & \Phi^{6}_{(\pm\frac{1}{2},\mp\frac{1}{2},-\frac{1}{2},-\frac{1}{2\sqrt{3}},+\frac{1}{\sqrt{6}})} \\
\Phi^{7}_{(\pm\frac{1}{2},\pm\frac{1}{2},-1,-\frac{1}{\sqrt{3}},-\frac{1}{\sqrt{6}})} & \Phi^{7}_{(\pm\frac{1}{2},\mp\frac{1}{2},-1,-\frac{1}{\sqrt{3}},-\frac{1}{\sqrt{6}})} & \Phi^{7}_{(\pm\frac{1}{2},\pm\frac{1}{2},+1,+\frac{1}{\sqrt{3}},+\frac{1}{\sqrt{6}})} & \Phi^{7}_{(\pm\frac{1}{2},\mp\frac{1}{2},+1,+\frac{1}{\sqrt{3}},+\frac{1}{\sqrt{6}})} \\
\Phi^{8}_{(\pm\frac{1}{2},\pm\frac{1}{2},+1,-\frac{1}{\sqrt{3}},-\frac{1}{\sqrt{6}})} & \Phi^{8}_{(\pm\frac{1}{2},\mp\frac{1}{2},+1,-\frac{1}{\sqrt{3}},-\frac{1}{\sqrt{6}})} & \Phi^{8}_{(\pm\frac{1}{2},\pm\frac{1}{2},-1,+\frac{1}{\sqrt{3}},+\frac{1}{\sqrt{6}})} & \Phi^{8}_{(\pm\frac{1}{2},\mp\frac{1}{2},-1,+\frac{1}{\sqrt{3}},+\frac{1}{\sqrt{6}})} \\
\Phi^{9}_{(\pm\frac{1}{2},\pm\frac{1}{2},\,0,+\frac{2}{\sqrt{3}},-\frac{1}{\sqrt{6}})} & \Phi^{9}_{(\pm\frac{1}{2},\mp\frac{1}{2},\,0,+\frac{2}{\sqrt{3}},-\frac{1}{\sqrt{6}})} & \Phi^{9}_{(\pm\frac{1}{2},\pm\frac{1}{2},\,0,-\frac{2}{\sqrt{3}},+\frac{1}{\sqrt{6}})} & \Phi^{9}_{(\pm\frac{1}{2},\mp\frac{1}{2},\,0,-\frac{2}{\sqrt{3}},+\frac{1}{\sqrt{6}})} \\
\Phi^{10}_{(\pm\frac{1}{2},\pm\frac{1}{2},\,0,\,0\,,+\frac{3}{\sqrt{6}})} & \Phi^{10}_{(\pm\frac{1}{2},\mp\frac{1}{2},\,0,\,0\,,+\frac{3}{\sqrt{6}})} & \Phi^{10}_{(\pm\frac{1}{2},\pm\frac{1}{2},\,0,\,0\,,-\frac{3}{\sqrt{6}})} & \Phi^{10}_{(\pm\frac{1}{2},\mp\frac{1}{2},\,0,\,0\,,-\frac{3}{\sqrt{6}})}
\end{array}
\tag{8.6}
$$

The values in the parenthesis indicate the $SU(2)_L \times SU(2)_R \times SU(4)$ weights. They are given in the order $(L_3, R_3, U_3, U_8, U_{15})$ respectively. The first two columns above show the $(2,2,\bar{10})$ Higgs fields and the last two columns show the $(2,2,10)$ Higgs fields. Again either up or down pairs in $(\pm\frac{1}{2}, \pm\frac{1}{2}\dots)$ and in $(\pm\frac{1}{2}, \mp\frac{1}{2}\dots)$ should be chosen. In the 210 Higgs matrix, the $(2,2,\bar{10})$ fields are located in the upper off diagonal block and the $(2,2,10)$ fields are located in the lower off diagonal block. Note that we haven't distinguished between them by means of any label.

The 10^{th} component of $(2,2,10)$ with (up,up) isospin and the 10^{th} component of $(2,2,\bar{10})$ with (down,down) isospin are $SU(3)_c \times U(1)_Q$ singlets. Indeed each of them could be in principal utilized to break the $SU(2)_L \times SU(2)_R \times SU(4)$ symmetry down to $SU(3)_c \times U(1)_Q$. But this spontaneous symmetry breaking initiates a very sudden transition between the final and initial states, resulting in extremely heavy gauge bosons with masses high above the Fermi scale. Since this is in conflict with the electroweak theory, we disregard these Higgs fields. From the other side the $(3,1,15)$, $(1,3,\bar{15})$ and $(1,1,15)$ Higgs fields are respectively defined as

$$
\begin{array}{lll}
\Phi^{1}_{(\Delta,0,+1,0,0)} & \Phi^{1}_{(0,\Delta,-1,0,0)} & \Phi^{1}_{(0,0,+1,0,0)} \\
\Phi^{2}_{(\Delta,0,+\frac{1}{2},+\frac{3}{2\sqrt{3}},0)} & \Phi^{2}_{(0,\Delta,-\frac{1}{2},-\frac{3}{2\sqrt{3}},0)} & \Phi^{2}_{(0,0,+\frac{1}{2},+\frac{3}{2\sqrt{3}},0)} \\
\Phi^{3}_{(\Delta,0,-\frac{1}{2},+\frac{3}{2\sqrt{3}},0)} & \Phi^{3}_{(0,\Delta,+\frac{1}{2},-\frac{3}{2\sqrt{3}},0)} & \Phi^{3}_{(0,0,-\frac{1}{2},+\frac{3}{2\sqrt{3}},0)} \\
\Phi^{4}_{(\Delta,0,-1,0,0)} & \Phi^{4}_{(0,\Delta,+1,0,0)} & \Phi^{4}_{(0,0,-1,0,0)} \\
\Phi^{5}_{(\Delta,0,-\frac{1}{2},-\frac{3}{2\sqrt{3}},0)} & \Phi^{5}_{(0,\Delta,+\frac{1}{2},+\frac{3}{2\sqrt{3}},0)} & \Phi^{5}_{(0,0,-\frac{1}{2},-\frac{3}{2\sqrt{3}},0)} \\
\Phi^{6}_{(\Delta,0,+\frac{1}{2},-\frac{3}{2\sqrt{3}},0)} & \Phi^{6}_{(0,\Delta,-\frac{1}{2},+\frac{3}{2\sqrt{3}},0)} & \Phi^{6}_{(0,0,+\frac{1}{2},-\frac{3}{2\sqrt{3}},0)} \\
\Phi^{7}_{(\Delta,0,0,0,0)} & \Phi^{7}_{(0,\Delta,0,0,0)} & \Phi^{7}_{(0,0,0,0,0)} \\
\Phi^{8}_{(\Delta,0,0,0,0)} & \Phi^{8}_{(0,\Delta,0,0,0)} & \Phi^{8}_{(0,0,0,0,0)} \\
\Phi^{9}_{(\Delta,0,+\frac{1}{2},+\frac{1}{2\sqrt{3}},+\frac{2}{\sqrt{6}})} & \Phi^{9}_{(0,\Delta,-\frac{1}{2},-\frac{1}{2\sqrt{3}},-\frac{2}{\sqrt{6}})} & \Phi^{9}_{(0,0,+\frac{1}{2},+\frac{1}{2\sqrt{3}},+\frac{2}{\sqrt{6}})} \\
\Phi^{10}_{(\Delta,0,-\frac{1}{2},+\frac{1}{2\sqrt{3}},+\frac{2}{\sqrt{6}})} & \Phi^{10}_{(0,\Delta,+\frac{1}{2},-\frac{1}{2\sqrt{3}},-\frac{2}{\sqrt{6}})} & \Phi^{10}_{(0,0,-\frac{1}{2},+\frac{1}{2\sqrt{3}},+\frac{2}{\sqrt{6}})} \\
\Phi^{11}_{(\Delta,0,0,-\frac{1}{\sqrt{3}},+\frac{2}{\sqrt{6}})} & \Phi^{11}_{(0,\Delta,0,+\frac{1}{\sqrt{3}},-\frac{2}{\sqrt{6}})} & \Phi^{11}_{(0,0,0,-\frac{1}{\sqrt{3}},+\frac{2}{\sqrt{6}})} \\
\Phi^{12}_{(\Delta,0,-\frac{1}{2},-\frac{1}{2\sqrt{3}},-\frac{2}{\sqrt{6}})} & \Phi^{12}_{(0,\Delta,+\frac{1}{2},+\frac{1}{2\sqrt{3}},+\frac{2}{\sqrt{6}})} & \Phi^{12}_{(0,0,-\frac{1}{2},-\frac{1}{2\sqrt{3}},-\frac{2}{\sqrt{6}})} \\
\Phi^{13}_{(\Delta,0,+\frac{1}{2},-\frac{1}{2\sqrt{3}},-\frac{2}{\sqrt{6}})} & \Phi^{13}_{(0,\Delta,-\frac{1}{2},+\frac{1}{2\sqrt{3}},+\frac{2}{\sqrt{6}})} & \Phi^{13}_{(0,0,+\frac{1}{2},-\frac{1}{2\sqrt{3}},-\frac{2}{\sqrt{6}})} \\
\Phi^{14}_{(\Delta,0,0,+\frac{1}{\sqrt{3}},-\frac{2}{\sqrt{6}})} & \Phi^{14}_{(0,\Delta,0,-\frac{1}{\sqrt{3}},+\frac{2}{\sqrt{6}})} & \Phi^{14}_{(0,0,0,+\frac{1}{\sqrt{3}},-\frac{2}{\sqrt{6}})} \\
\Phi^{15}_{(\Delta,0,0,0,0)} & \Phi^{15}_{(0,\Delta,0,0,0)} & \Phi^{15}_{(0,0,0,0,0)}
\end{array}
\tag{8.7}
$$

The Higgs fields of the $(1,1,15)$ multiplet coincides with those of the $(3,1,15)$ and $(1,3,\bar{15})$ multiplets as seen from Ω_{11}. The last Higgs field of the $(1,1,15)$ multiplet above which is the Φ^{15}_{00} decomposes under $SU(2)_L \times SU(2)_R \times SU(3)_c \times U(1)_{B-L}$ as a $(1,1,1)_0$ singlet. It could be utilized to break the $SU(2)_L \times SU(2)_R \times SU(4)$ symmetry down to $SU(2)_L \times SU(2)_R \times SU(3)_c \times U(1)_{B-L}$. We will use instead a similar singlet from the 45 representation which we first introduce in § 9. A discussion about the necessity of using such a singlet in the Higgs mechanism will be surveyed in § 11. The Higgs fields of the $(2, 2, 6)$ multiplet are

$$\begin{array}{ll}
\Phi^1_{(\pm\frac{1}{2},\pm\frac{1}{2},\ 0\ ,+\frac{1}{\sqrt{3}}\ ,+\frac{1}{\sqrt{6}})} & \Phi^1_{(\pm\frac{1}{2},\mp\frac{1}{2},\ 0\ ,+\frac{1}{\sqrt{3}}\ ,+\frac{1}{\sqrt{6}})} \\
\Phi^2_{(\pm\frac{1}{2},\pm\frac{1}{2},+\frac{1}{2},-\frac{1}{2\sqrt{3}},+\frac{1}{\sqrt{6}})} & \Phi^2_{(\pm\frac{1}{2},\mp\frac{1}{2},+\frac{1}{2},-\frac{1}{2\sqrt{3}},+\frac{1}{\sqrt{6}})} \\
\Phi^3_{(\pm\frac{1}{2},\pm\frac{1}{2},+\frac{1}{2},+\frac{1}{2\sqrt{3}},-\frac{1}{\sqrt{6}})} & \Phi^3_{(\pm\frac{1}{2},\mp\frac{1}{2},+\frac{1}{2},+\frac{1}{2\sqrt{3}},-\frac{1}{\sqrt{6}})} \\
\Phi^4_{(\pm\frac{1}{2},\pm\frac{1}{2},\ 0\ ,-\frac{1}{\sqrt{3}}\ ,-\frac{1}{\sqrt{6}})} & \Phi^4_{(\pm\frac{1}{2},\mp\frac{1}{2},\ 0\ ,-\frac{1}{\sqrt{3}}\ ,-\frac{1}{\sqrt{6}})} \\
\Phi^5_{(\pm\frac{1}{2},\pm\frac{1}{2},-\frac{1}{2},+\frac{1}{2\sqrt{3}},-\frac{1}{\sqrt{6}})} & \Phi^5_{(\pm\frac{1}{2},\mp\frac{1}{2},-\frac{1}{2},+\frac{1}{2\sqrt{3}},-\frac{1}{\sqrt{6}})} \\
\Phi^6_{(\pm\frac{1}{2},\pm\frac{1}{2},-\frac{1}{2},-\frac{1}{2\sqrt{3}},+\frac{1}{\sqrt{6}})} & \Phi^6_{(\pm\frac{1}{2},\mp\frac{1}{2},-\frac{1}{2},-\frac{1}{2\sqrt{3}},+\frac{1}{\sqrt{6}})}
\end{array} \tag{8.8}$$

These Higgs fields are overlapping with those of the $(2, 2, 10)$ and the $(2, 2, \bar{1}0)$ multiplets. All of the $(2, 2, 6)$ Higgs fields carry $SU(3)_c$ color and fractional electric charges. Therefore none of them can be used in the Higgs mechanism. Finally we have a $SU(2)_L \times SU(2)_R \times SU(4)$ singlet which is

$$\Phi^s_{(0,0,0,0,0)} \tag{8.9}$$

This $(1, 1, 1)$ singlet was shortly denoted with Φ^s_{00} in the matrix. For the Higgs mechanism we will use the following Higgs term

$$\Phi_{111} \equiv \Phi^s_{00}\mathbf{\Gamma}^s_{00} = \frac{1}{\sqrt{32}}\phi_{78910}\ \ \Gamma_7\Gamma_8\Gamma_9\Gamma_{10}\ ; \qquad Tr\left[(\Phi_{111})^2\right] = (\Phi^s_{00})^2 \tag{8.10}$$

where $(\ )^2 = (\ )(\ )^\dagger$.

8.1 *Features of the* Φ^s_{00}

(i) Again we consider the most general case in that we break the $SO(10)$ gauge group with the Higgs field Φ^s_{00}. Then the vev of this Higgs fields endows the gauge fields of the $(2, 2, 6)$ multiplet with mass.

(ii) The Φ^s_{00} singlet of 210 can not be employed in the Yukawa sector. Because it fails to produce dirac mass terms for fermions. This fact applies to all Higgs fields in the 210 Higgs representation. This can also be seen from the matrix of Ω's. It couples Ψ_L with $\bar{\Psi}_L$ and not Ψ_L with $\bar{\Psi}_R$.

9. THE HIGGS MULTIPLETS : THE (45)-REPRESENTATION

9.1 The Structure

The 45 is the collection of antisymmetric objects that can be constructed from the bi-products of the Γ_i basis with $i = (1 \ldots 10)$ so that $\Sigma_{ab} = -\Sigma_{ba}$. These are the generators Σ_{ab} of $SO(10)$ and were introduced previously in eq. (2.12). In order to investigate the 45 Higgs representation, we define 45 real scalar fields ϕ_{ab} where $a, b = (1, \ldots, 10)$ with $\phi_{ab} = -\phi_{ba}$ and construct the sum

$$\frac{1}{2\sqrt{2}} \Sigma_{ab}\, \phi_{ab} = \frac{1}{2}\, \mathbf{\Gamma} \cdot \Phi \;\; ; \qquad \frac{1}{8} Tr \left[(\Sigma_{ab}\, \phi_{ab})^2 \right] = \sum_{a,b=1}^{10} \phi_{ab}^2, \quad a \neq b \tag{9.1}$$

which is alike the gauge term except for that we have scalar fields instead of vector fields. The Higgs fields Φ are complex and are spanned by the basis of ϕ_{ab}. The $\mathbf{\Gamma}$'s are made of the bi-products $\Gamma_i \Gamma_j$ with $i \neq j$. The decomposition of the 45 Higgs fields is akin to that of the 45 gauge bosons studied in § 3 [53]. The decomposition of the 45 in terms of the maximal subgroup $SU(4) \times SU(2)_L \times SU(2)_R$ and in terms of its descent $SU(3)_c \times SU(2)_L \times SU(2)_R \times U(1)_{B-L}$ is done as

$$\begin{array}{lcllllllllll}
45 & = & (1,1,15) & + & (2,2,6) & + & (1,3,1) & + & (1,1,3) & & \\
45 & = & (1,1,8)_0 & + & (1,1,3)_{2/3} & + & (1,1,\bar{3})_{-2/3} & + & (1,1,1)_0 & + & (2,2,\bar{3})_{2/3} \\
 & & (2,2,3)_{-2/3} & + & (1,3,1)_0 & + & (1,1,3)_0 & & & &
\end{array} \tag{9.2}$$

respectively, where the subscripts denote $B - L$ numbers. The expressions for the Higgs fields that fall into the above multiplets in terms of the basis ϕ_{ab} can be easily recovered from our previous analysis in § 3. The $(1, 1, 15)$ members are

$$\begin{array}{llllll}
\bar{\Phi}^4_{00} = \Phi^1_{00} = & (\phi_{45} + \phi_{63} + i\,\phi_{53} + i\,\phi_{46})/2 & \Phi^9_{00} & = & (\phi_{23} + \phi_{14} + i\phi_{31} + i\phi_{24})/2 \\
\bar{\Phi}^5_{00} = \Phi^2_{00} = & (\phi_{52} + \phi_{61} + i\,\phi_{62} + i\,\phi_{15})/2 & \Phi^{10}_{00} & = & (\phi_{25} + \phi_{61} + i\phi_{51} + i\phi_{62})/2 \\
\bar{\Phi}^6_{00} = \Phi^3_{00} = & (\phi_{23} + \phi_{41} + i\,\phi_{31} + i\,\phi_{42})/2 & \Phi^{11}_{00} & = & (\phi_{45} + \phi_{63} + i\phi_{53} + i\phi_{64})/2 \\
\bar{\Phi}^7_{00} = \Phi^7_{00} = & (\phi_{21} + \phi_{43} + 2\,\phi_{65})/\sqrt{6} & \Phi^{15}_{00} & = & (\phi_{21} + \phi_{43} - \phi_{65})/\sqrt{3} \\
\bar{\Phi}^8_{00} = \Phi^8_{00} = & (\phi_{21} - 2\,\phi_{43} - \phi_{65})/\sqrt{6} & & &
\end{array} \tag{9.3}$$

where Φ^1_{00}, Φ^2_{00} and Φ^3_{00} are conjugated to $\bar{\Phi}^4_{00}, \bar{\Phi}^5_{00}$ and $\bar{\Phi}^6_{00}$ respectively. Φ^7_{00} and Φ^8_{00} are made of diagonal elements. Also Φ^{12}_{00}, Φ^{13}_{00}, Φ^{14}_{00} are conjugated to Φ^9_{00}, Φ^{10}_{00}, Φ^{11}_{00}. The two zeros in the subscripts show left and right isospin states. The $(3, 1, 1)$ and the $(1, 3, 1)$ triplet fields are respectively

$$\begin{array}{llllll}
\Phi^1_{\uparrow 0} & = & (\phi_{98} + \phi_{107} + i\phi_{79} + i\phi_{108})/2 & \Phi^1_{0\uparrow} & = & (\phi_{98} + \phi_{710} + i\phi_{79} + i\phi_{810})/2 \\
\Phi^3_{\hat{0}0} & = & (\phi_{87} + \phi_{109})/\sqrt{2} & \Phi^3_{0\hat{0}} & = & (\phi_{87} + \phi_{910})/\sqrt{2} \\
\Phi^2_{\downarrow 0} & = & (\phi_{98} + \phi_{107} - i\phi_{79} - i\phi_{108})/2 & \Phi^2_{0\downarrow} & = & (\phi_{98} + \phi_{710} - i\phi_{79} - i\phi_{810})/2
\end{array} \tag{9.4}$$

The subscripts $(\uparrow, \hat{0}, \downarrow)$ show left and right isospin states of the respective isospin triplet. The hat on the zero is, as done before, introduced to distinguish between the isospin neutral fields. The fields in the coset of the maximal subgroup and the $SO(10)$ group, fall into the $(2, 2, 6)$ multiplet. These are

$$\begin{array}{llllll}
\Phi^1_{\uparrow\uparrow} & = & (\phi_{75} + \phi_{68} + i\phi_{76} + i\phi_{85})/2 & \Phi^{13}_{\downarrow\uparrow} & = & (\phi_{95} + \phi_{610} + i\phi_{96} + i\phi_{105})/2 \\
\Phi^2_{\uparrow\uparrow} & = & (\phi_{37} + \phi_{48} + i\phi_{74} + i\phi_{38})/2 & \Phi^{14}_{\downarrow\uparrow} & = & (\phi_{39} + \phi_{410} + i\phi_{94} + i\phi_{310})/2 \\
\Phi^3_{\uparrow\uparrow} & = & (\phi_{71} + \phi_{82} + i\phi_{27} + i\phi_{81})/2 & \Phi^{15}_{\downarrow\uparrow} & = & (\phi_{91} + \phi_{102} + i\phi_{29} + i\phi_{101})/2 \\
 & & & & & \\
\Phi^4_{\uparrow\downarrow} & = & (\phi_{59} + \phi_{610} + i\phi_{69} + i\phi_{105})/2 & \Phi^{16}_{\downarrow\downarrow} & = & (\phi_{75} + \phi_{86} + i\phi_{76} + i\phi_{58})/2 \\
\Phi^5_{\uparrow\downarrow} & = & (\phi_{93} + \phi_{410} + i\phi_{49} + i\phi_{310})/2 & \Phi^{17}_{\downarrow\downarrow} & = & (\phi_{37} + \phi_{84} + i\phi_{74} + i\phi_{83})/2 \\
\Phi^6_{\uparrow\downarrow} & = & (\phi_{19} + \phi_{102} + i\phi_{92} + i\phi_{101})/2 & \Phi^{18}_{\downarrow\downarrow} & = & (\phi_{71} + \phi_{28} + i\phi_{27} + i\phi_{18})/2
\end{array} \tag{9.5}$$

The subscripts $\uparrow\downarrow$ etc. show left and right isospin states. The above fields are ordered as below, where the values in parentheses denote weights with respect to the maximal subgroup in the order L_3, R_3, U_3, U_8 and U_{15}. We have

$$
\left.\begin{array}{l}
\Phi^1_{(0,0,+1,0,0)} \\
\Phi^2_{(0,0,+\frac{1}{2},+\frac{3}{2\sqrt{3}},0)} \\
\Phi^3_{(0,0,-\frac{1}{2},+\frac{3}{2\sqrt{3}},0)} \\
\Phi^4_{(0,0,-1,0,0)} \\
\Phi^5_{(0,0,-\frac{1}{2},-\frac{3}{2\sqrt{3}},0)} \\
\Phi^6_{(0,0,+\frac{1}{2},-\frac{3}{2\sqrt{3}},0)} \\
\Phi^7_{(0,0,0,0,0)} \\
\Phi^8_{(0,0,0,0,0)} \\
\Phi^9_{(0,0,+\frac{1}{2},+\frac{1}{2\sqrt{3}},+\frac{2}{\sqrt{6}})} \\
\Phi^{10}_{(0,0,-\frac{1}{2},+\frac{1}{2\sqrt{3}},+\frac{2}{\sqrt{6}})} \\
\Phi^{11}_{(0,0,0,-\frac{1}{\sqrt{3}},+\frac{2}{\sqrt{6}})} \\
\Phi^{12}_{(0,0,-\frac{1}{2},-\frac{1}{2\sqrt{3}},-\frac{2}{\sqrt{6}})} \\
\Phi^{13}_{(0,0,+\frac{1}{2},-\frac{1}{2\sqrt{3}},-\frac{2}{\sqrt{6}})} \\
\Phi^{14}_{(0,0,0,+\frac{1}{\sqrt{3}},-\frac{2}{\sqrt{6}})} \\
\Phi^{15}_{(0,0,0,0,0)}
\end{array}\right\}(1,1,15)
\quad
\left.\begin{array}{l}
\Phi^1_{(+\frac{1}{2},+\frac{1}{2},+\frac{1}{2},+\frac{1}{2\sqrt{3}},-\frac{1}{\sqrt{6}})} \\
\Phi^2_{(+\frac{1}{2},+\frac{1}{2},-\frac{1}{2},+\frac{1}{2\sqrt{3}},-\frac{1}{\sqrt{6}})} \\
\Phi^3_{(+\frac{1}{2},+\frac{1}{2},\,0\,,-\frac{1}{\sqrt{3}},-\frac{1}{\sqrt{6}})} \\
\\
\Phi^4_{(+\frac{1}{2},-\frac{1}{2},+\frac{1}{2},+\frac{1}{2\sqrt{3}},-\frac{1}{\sqrt{6}})} \\
\Phi^5_{(+\frac{1}{2},-\frac{1}{2},-\frac{1}{2},+\frac{1}{2\sqrt{3}},-\frac{1}{\sqrt{6}})} \\
\Phi^6_{(+\frac{1}{2},-\frac{1}{2},\,0\,,-\frac{1}{\sqrt{3}},-\frac{1}{\sqrt{6}})} \\
\\
\Phi^{13}_{(-\frac{1}{2},+\frac{1}{2},+\frac{1}{2},+\frac{1}{2\sqrt{3}},-\frac{1}{\sqrt{6}})} \\
\Phi^{14}_{(-\frac{1}{2},+\frac{1}{2},-\frac{1}{2},+\frac{1}{2\sqrt{3}},-\frac{1}{\sqrt{6}})} \\
\Phi^{15}_{(-\frac{1}{2},+\frac{1}{2},\,0\,,-\frac{1}{\sqrt{3}},-\frac{1}{\sqrt{6}})} \\
\\
\Phi^{16}_{(-\frac{1}{2},-\frac{1}{2},+\frac{1}{2},+\frac{1}{2\sqrt{3}},-\frac{1}{\sqrt{6}})} \\
\Phi^{17}_{(-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},+\frac{1}{2\sqrt{3}},-\frac{1}{\sqrt{6}})} \\
\Phi^{18}_{(-\frac{1}{2},-\frac{1}{2},\,0\,,-\frac{1}{\sqrt{3}},-\frac{1}{\sqrt{6}})}
\end{array}\right\}(2,2,3)
\quad
\left.\begin{array}{l}
\Phi^7_{(-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2\sqrt{3}},+\frac{1}{\sqrt{6}})} \\
\Phi^8_{(-\frac{1}{2},-\frac{1}{2},+\frac{1}{2},-\frac{1}{2\sqrt{3}},+\frac{1}{\sqrt{6}})} \\
\Phi^9_{(-\frac{1}{2},-\frac{1}{2},\,0\,,+\frac{1}{\sqrt{3}},+\frac{1}{\sqrt{6}})} \\
\\
\Phi^{10}_{(-\frac{1}{2},+\frac{1}{2},-\frac{1}{2},-\frac{1}{2\sqrt{3}},+\frac{1}{\sqrt{6}})} \\
\Phi^{11}_{(-\frac{1}{2},+\frac{1}{2},+\frac{1}{2},-\frac{1}{2\sqrt{3}},+\frac{1}{\sqrt{6}})} \\
\Phi^{12}_{(-\frac{1}{2},+\frac{1}{2},\,0\,,+\frac{1}{\sqrt{3}},+\frac{1}{\sqrt{6}})} \\
\\
\Phi^{19}_{(+\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2\sqrt{3}},+\frac{1}{\sqrt{6}})} \\
\Phi^{20}_{(+\frac{1}{2},-\frac{1}{2},+\frac{1}{2},-\frac{1}{2\sqrt{3}},+\frac{1}{\sqrt{6}})} \\
\Phi^{21}_{(+\frac{1}{2},-\frac{1}{2},\,0\,,+\frac{1}{\sqrt{3}},+\frac{1}{\sqrt{6}})} \\
\\
\Phi^{22}_{(+\frac{1}{2},+\frac{1}{2},-\frac{1}{2},-\frac{1}{2\sqrt{3}},+\frac{1}{\sqrt{6}})} \\
\Phi^{23}_{(+\frac{1}{2},+\frac{1}{2},+\frac{1}{2},-\frac{1}{2\sqrt{3}},+\frac{1}{\sqrt{6}})} \\
\Phi^{24}_{(+\frac{1}{2},+\frac{1}{2},\,0\,,+\frac{1}{\sqrt{3}},+\frac{1}{\sqrt{6}})}
\end{array}\right\}(2,2,\bar{3})
$$

$$
\left.\begin{array}{l}
\Phi^1_{(\uparrow,0,0,0,0)} \\
\Phi^3_{(\hat{0},0,0,0,0)} \\
\Phi^2_{(\downarrow,0,0,0,0)}
\end{array}\right\}(3,1,1)
\qquad
\left.\begin{array}{l}
\Phi^1_{(0,\uparrow,0,0,0)} \\
\Phi^3_{(0,\hat{0},0,0,0)} \\
\Phi^2_{(0,\downarrow,0,0,0)}
\end{array}\right\}(1,3,1)
\tag{9.6}
$$

The weight diagrams of these scalar fields are the same with those illustrated in Fig. (3.1). The Higgs fields of the $(1,1,8)$ form a color octet and are electrically neutral. The $(2,2,3)_{-2/3}$ and its conjugate $(2,2,\bar{3})_{2/3}$ are color triplets and color anti-triplets respectively, which carry electric charges. Similarly the $(1,1,3)_{2/3}$ and its conjugate $(1,1,\bar{3})_{-2/3}$ are also color triplets with electric charge. None of these fields should be given any vev.

However the $(1,3,1)_0$ is a right isospin triplet, color singlet. These Higgs triplet comes with 3 different electric charges, namely $(+1,0,-1)$ and the $(3,1,1)_0$ is a left isospin triplet with electric charges $(+1,0,-1)$. Since the neutral members of both triplets are $SU(3)\times U(1)_Q$ singlets, they can be candidates for the Higgs mechanism. The neutral Higgs field of the former triplet breaks $SU(2)_R$ down to $U(1)_R$ and the neutral Higgs field of the latter triplet breaks $SU(2)_L$ down to $U(1)_L$. Furthermore none of the two can generate Dirac masses for fermions.

Finally the $(1,1,1)_0$ is a singlet under the $SU(3)\times U(1)_Q$ symmetry. It is a candidate for the Higgs mechanism and breaks the following initial symmetries to the final symmetries

(i) $SU(4)\times SU(2)_L\times SU(2)_R$ down to $SU(3)\times SU(2)_L\times SU(2)_R\times U(1)_{B-L}$.

(ii) $SU(4)\times U(1)_{L+R}$ down to $SU(3)\times U(1)_{L+R}\times U(1)_{B-L}$.

These descents are illustrated in Fig. (11.1) together with the Higgs multiplets studied in the preceding sections.

Using the isomorphism between $SO(6)\times SO(4)$ and $SU(4)\times SU(2)\times SU(2)$ we can rewrite the expansion in eq. (9.1) in terms of the unitary generators given in eq. (3.9) and (3.10). The $(1,1,1)_0$ singlet is nothing but the Φ^{15}_{00} Higgs field in eq. (9.3) and multiplies $\mathbf{\Gamma^{15}_{00}}$ in the isomorphically equivalent expansion which is the U_{15} generator. We denote the sum in eq. (9.1) formally as ϕ. Then ϕ transforms under the representation of the maximal subgroup such that

$$\phi \rightarrow \Lambda^\dagger \phi\, \Lambda \quad ; \quad \Lambda = \exp\left(-igU_i\lambda_i - igL_i\lambda'_i - igR_i\lambda''_i\right) \tag{9.7}$$

where the λ's are some parameters. Actually the S_i basis for $i=(1,\ldots 24)$ should also be included in the exponential so that the transformation is generalized to the $SO(10)$ case. We can expand Λ around the identity and look at the infinitesimal transformation of ϕ. From the expansion we get

$$\phi \rightarrow \phi + ig\left[\phi, U_i\right]\lambda_i + ig\left[\phi, R_i\right]\lambda''_i + ig\left[\phi, L_i\right]\lambda'_i + O(g^2) + \ldots \tag{9.8}$$

The higher order terms can be found through the Campbell-Baker-Hausdorff lemma. In general ϕ, prior to some spontaneous breakdown, transforms not only with respect to Λ but also separately under each of the exponentials.

Since the generators of each gauge group in the direct product of the maximal subgroup mutually commute, they can be collected in a single exponential as above which is sufficiently general.

The vacuum is by definition the *stable* ground state of the Higgs field. If ϕ receives spontaneously a vev, some of the commutators above will not vanish any more, i.e., the initial symmetry will be lost. The residual symmetry that the vacuum respects can be sorted out from the commutators in the expansion. Let us assume that $\phi_{vac} = \langle \Phi^{15}_{00} \rangle \cdot \mathbf{\Gamma}^{15}_{00}$, where $\langle \Phi^{15}_{00} \rangle$ denotes the vev received by the $(1,1,1)_0$ singlet. Then we obtain

$$\begin{array}{lcllll} [\,L_i, \phi_{vac}] & = & 0 & i = 1, \ldots, 3 & ; & SU(2)_L \\ [\,R_i, \phi_{vac}] & = & 0 & i = 1, \ldots, 3 & ; & SU(2)_R \\ [\,U_i, \phi_{vac}] & = & 0 & i = 1, \ldots, 8 & ; & SU(3) \\ [\,U_i, \phi_{vac}] & = & 0 & i = 15 & ; & U(1)_{B-L} \\ [\,U_i, \phi_{vac}] & \neq & 0 & i = 9, \ldots, 14 & ; & X \text{ bosons} \\ [\,S_i, \phi_{vac}] & \neq & 0 & i = 1, \ldots, 24 & ; & (2,2,6) \text{ bosons} \end{array} \tag{9.9}$$

This choice of ϕ_{vac} above lets the vacuum to develop a minimum that is invariant under the residual $SU(3)_c \times SU(2)_L \times SU(2)_R \times U(1)_{B-L}$ gauge symmetry. It is seen from the last line above that the transformations generated by U_i for $i = (9, \ldots, 14)$ and S_i under $i = (1, \ldots, 24)$ are no more symmetries of the vacuum. Indeed these symmetries can be recovered through the *unitary gauge* at the minima, in which the X_α and $(2,2,6)$ gauge bosons eat the goldstone bosons. We have occasionally expressed the above commutations shortly as

$$U_{15} = \mathbf{\Gamma}^{15}_{00} \quad \rightarrow \quad \Delta_R = \Delta_L = \Delta_{B-L} = 0 \quad \Rightarrow \quad \Delta_Q = 0 \tag{9.10}$$

Here Δ's denote the amount of non-commutation with respect to the corresponding symmetries. Note that we have not devised any potential $V(\phi)$ yet. This will be first done in conjunction with many other multiplets in § 11.1. The term suitable for the Higgs mechanism will be chosen as

$$\Phi_{45} = \frac{1}{2}\, \Phi^{15}_{00}\, \mathbf{\Gamma}^{15}_{00} + \text{Goldstone modes} \tag{9.11}$$

Since the residual symmetry of the vacuum after spontaneous symmetry breakdown is the $SU(3)_c \times U(1)_Q$ there will exist $45 - 8 - 1 = 36$ massive gauge fields in $SO(10)$. Therefore we need 36 Goldstone modes. We choose them out of the 45 Higgs multiplet as

$$\begin{aligned} \text{Goldstone modes} =& \frac{1}{2}\, \{ \Phi^1_{\uparrow\uparrow}\, \mathbf{\Gamma}^1_{\uparrow\uparrow} + \Phi^2_{\uparrow\uparrow}\, \mathbf{\Gamma}^2_{\uparrow\uparrow} + \Phi^3_{\uparrow\uparrow}\, \mathbf{\Gamma}^3_{\uparrow\uparrow} + \Phi^4_{\uparrow\downarrow}\, \mathbf{\Gamma}^4_{\uparrow\downarrow} + \Phi^5_{\uparrow\downarrow}\, \mathbf{\Gamma}^5_{\uparrow\downarrow} + \Phi^6_{\uparrow\downarrow}\, \mathbf{\Gamma}^6_{\downarrow\downarrow} + \\ & \Phi^7_{\downarrow\downarrow}\, \mathbf{\Gamma}^7_{\downarrow\downarrow} + \Phi^8_{\downarrow\downarrow}\, \mathbf{\Gamma}^8_{\downarrow\downarrow} + \Phi^9_{\downarrow\downarrow}\, \mathbf{\Gamma}^9_{\downarrow\downarrow} + \Phi^{10}_{\downarrow\uparrow}\, \mathbf{\Gamma}^{10}_{\downarrow\uparrow} + \Phi^{11}_{\downarrow\uparrow}\, \mathbf{\Gamma}^{11}_{\downarrow\uparrow} + \Phi^{12}_{\downarrow\uparrow}\, \mathbf{\Gamma}^{12}_{\downarrow\uparrow} + \\ & \Phi^{13}_{\downarrow\uparrow}\, \mathbf{\Gamma}^{13}_{\downarrow\uparrow} + \Phi^{14}_{\downarrow\uparrow}\, \mathbf{\Gamma}^{14}_{\downarrow\uparrow} + \Phi^{15}_{\downarrow\uparrow}\, \mathbf{\Gamma}^{15}_{\downarrow\uparrow} + \Phi^{16}_{\downarrow\downarrow}\, \mathbf{\Gamma}^{16}_{\downarrow\downarrow} + \Phi^{17}_{\downarrow\downarrow}\, \mathbf{\Gamma}^{17}_{\downarrow\downarrow} + \Phi^{18}_{\downarrow\downarrow}\, \mathbf{\Gamma}^{18}_{\downarrow\downarrow} + \\ & \Phi^{19}_{\uparrow\downarrow}\, \mathbf{\Gamma}^{19}_{\uparrow\downarrow} + \Phi^{20}_{\uparrow\downarrow}\, \mathbf{\Gamma}^{20}_{\uparrow\downarrow} + \Phi^{21}_{\uparrow\downarrow}\, \mathbf{\Gamma}^{21}_{\uparrow\uparrow} + \Phi^{22}_{\uparrow\uparrow}\, \mathbf{\Gamma}^{22}_{\uparrow\downarrow} + \Phi^{23}_{\uparrow\uparrow}\, \mathbf{\Gamma}^{23}_{\uparrow\uparrow} + \Phi^{24}_{\uparrow\uparrow}\, \mathbf{\Gamma}^{24}_{\uparrow\uparrow} + \\ & \Phi^9_{00}\, \mathbf{\Gamma}^9_{00} + \Phi^{10}_{00}\, \mathbf{\Gamma}^{10}_{00} + \Phi^{11}_{00}\, \mathbf{\Gamma}^{11}_{00} + \Phi^{12}_{00}\, \mathbf{\Gamma}^{12}_{00} + \Phi^{13}_{00}\, \mathbf{\Gamma}^{13}_{00} + \Phi^{14}_{00}\, \mathbf{\Gamma}^{14}_{00} + \\ & \Phi^1_{\uparrow 0}\, \mathbf{\Gamma}^1_{\uparrow 0} + \Phi^3_{\hat{0}0}\, \mathbf{\Gamma}^3_{\hat{0}0} + \Phi^2_{\downarrow 0}\, \mathbf{\Gamma}^2_{\downarrow 0} + \Phi^1_{0\uparrow}\, \mathbf{\Gamma}^1_{0\uparrow} + \Phi^3_{0\hat{0}}\, \mathbf{\Gamma}^3_{0\hat{0}} + \Phi^2_{0\downarrow}\, \mathbf{\Gamma}^2_{0\downarrow} \} \end{aligned} \tag{9.12}$$

These Higgs fields will receive no vev but will provide 36 longitudinal degrees of freedom for other gauge fields becoming massive in $SO(10)$. They can be gauged away via a unitary gauge in the 45 representation. We will come back to this point later in § 11. Note that the $\mathbf{\Gamma}$'s in the above expansion are labelled in the same way as the Higgs fields of the 45 and can be easily recognized. We have

$$\begin{array}{lll} \left(\mathbf{\Gamma}^7_{\downarrow\downarrow}\right)^\dagger = \mathbf{\Gamma}^1_{\uparrow\uparrow} = & (\Sigma_{75} + \Sigma_{68} - i\,\Sigma_{76} - i\,\Sigma_{85})/4, & \left(\mathbf{\Gamma}^{19}_{\uparrow\downarrow}\right)^\dagger = \mathbf{\Gamma}^{13}_{\downarrow\uparrow} = (\Sigma_{95} + \Sigma_{610} - i\,\Sigma_{96} - i\,\Sigma_{105})/4 \\ \left(\mathbf{\Gamma}^8_{\downarrow\downarrow}\right)^\dagger = \mathbf{\Gamma}^2_{\uparrow\uparrow} = & (\Sigma_{37} + \Sigma_{48} - i\,\Sigma_{74} - i\,\Sigma_{38})/4, & \left(\mathbf{\Gamma}^{20}_{\uparrow\downarrow}\right)^\dagger = \mathbf{\Gamma}^{14}_{\downarrow\uparrow} = (\Sigma_{39} + \Sigma_{410} - i\,\Sigma_{94} - i\,\Sigma_{310})/4 \\ \left(\mathbf{\Gamma}^9_{\downarrow\downarrow}\right)^\dagger = \mathbf{\Gamma}^3_{\uparrow\uparrow} = & (\Sigma_{71} + \Sigma_{82} - i\,\Sigma_{27} - i\,\Sigma_{81})/4, & \left(\mathbf{\Gamma}^{21}_{\uparrow\downarrow}\right)^\dagger = \mathbf{\Gamma}^{15}_{\downarrow\uparrow} = (\Sigma_{91} + \Sigma_{102} - i\,\Sigma_{29} - i\,\Sigma_{101})/4 \end{array}$$

$$\begin{array}{lll} \left(\mathbf{\Gamma}^{10}_{\downarrow\uparrow}\right)^\dagger = \mathbf{\Gamma}^4_{\uparrow\downarrow} = & (\Sigma_{59} + \Sigma_{610} - i\,\Sigma_{69} - i\,\Sigma_{105})/4, & \left(\mathbf{\Gamma}^{22}_{\uparrow\uparrow}\right)^\dagger = \mathbf{\Gamma}^{16}_{\downarrow\downarrow} = (\Sigma_{75} + \Sigma_{86} - i\,\Sigma_{76} - i\,\Sigma_{58})/4 \\ \left(\mathbf{\Gamma}^{11}_{\downarrow\uparrow}\right)^\dagger = \mathbf{\Gamma}^5_{\uparrow\downarrow} = & (\Sigma_{93} + \Sigma_{410} - i\,\Sigma_{49} - i\,\Sigma_{310})/4, & \left(\mathbf{\Gamma}^{23}_{\uparrow\uparrow}\right)^\dagger = \mathbf{\Gamma}^{17}_{\downarrow\downarrow} = (\Sigma_{37} + \Sigma_{84} - i\,\Sigma_{74} - i\,\Sigma_{83})/4 \\ \left(\mathbf{\Gamma}^{12}_{\downarrow\uparrow}\right)^\dagger = \mathbf{\Gamma}^6_{\uparrow\downarrow} = & (\Sigma_{19} + \Sigma_{102} - i\,\Sigma_{92} - i\,\Sigma_{101})/4, & \left(\mathbf{\Gamma}^{24}_{\uparrow\uparrow}\right)^\dagger = \mathbf{\Gamma}^{18}_{\downarrow\downarrow} = (\Sigma_{71} + \Sigma_{28} - i\,\Sigma_{27} - i\,\Sigma_{18})/4 \end{array}$$

$$
\begin{array}{rclcl}
\left(\boldsymbol{\Gamma}^4_{00}\right)^\dagger & = & \boldsymbol{\Gamma}^1_{00} & = & (\Sigma_{45}+\Sigma_{63}-i\,\Sigma_{53}-i\,\Sigma_{46})/4 \\
\left(\boldsymbol{\Gamma}^5_{00}\right)^\dagger & = & \boldsymbol{\Gamma}^2_{00} & = & (\Sigma_{52}+\Sigma_{61}-i\,\Sigma_{62}-i\,\Sigma_{15})/4 \\
\left(\boldsymbol{\Gamma}^6_{00}\right)^\dagger & = & \boldsymbol{\Gamma}^3_{00} & = & (\Sigma_{23}+\Sigma_{41}-i\,\Sigma_{31}-i\,\Sigma_{42})/4 \\
\left(\boldsymbol{\Gamma}^7_{00}\right)^\dagger & = & \boldsymbol{\Gamma}^7_{00} & = & (\Sigma_{21}+\Sigma_{65})/\sqrt{6} \\
\left(\boldsymbol{\Gamma}^8_{00}\right)^\dagger & = & \boldsymbol{\Gamma}^8_{00} & = & (\Sigma_{21}-\Sigma_{43})/\sqrt{6} \\
\left(\boldsymbol{\Gamma}^{12}_{00}\right)^\dagger & = & \boldsymbol{\Gamma}^9_{00} & = & (\Sigma_{23}+\Sigma_{14}-i\,\Sigma_{31}-i\,\Sigma_{24})/4 \\
\left(\boldsymbol{\Gamma}^{13}_{00}\right)^\dagger & = & \boldsymbol{\Gamma}^{10}_{00} & = & (\Sigma_{25}+\Sigma_{61}-i\,\Sigma_{51}-i\,\Sigma_{62})/4 \\
\left(\boldsymbol{\Gamma}^{14}_{00}\right)^\dagger & = & \boldsymbol{\Gamma}^{11}_{00} & = & (\Sigma_{45}+\Sigma_{63}-i\,\Sigma_{53}-i\,\Sigma_{64})/4 \\
 & & \boldsymbol{\Gamma}^{15}_{00} & = & (\Sigma_{21}+\Sigma_{43}-\Sigma_{65})/\sqrt{12}
\end{array}
\tag{9.13}
$$

$$
\begin{array}{rcl}
\boldsymbol{\Gamma}^1_{\uparrow 0} & = & (\Sigma_{98}+\Sigma_{107}-i\,\Sigma_{79}-i\,\Sigma_{108})/4 \\
\boldsymbol{\Gamma}^3_{\hat{0}0} & = & (\Sigma_{87}+\Sigma_{109})/\sqrt{8} \\
\boldsymbol{\Gamma}^2_{\downarrow 0} & = & (\Sigma_{98}+\Sigma_{107}+i\,\Sigma_{79}+i\,\Sigma_{108})/4 \\
 & & \\
\boldsymbol{\Gamma}^1_{0\uparrow} & = & (\Sigma_{98}+\Sigma_{710}-i\,\Sigma_{79}-i\,\Sigma_{810})/4 \\
\boldsymbol{\Gamma}^3_{0\hat{0}} & = & (\Sigma_{87}+\Sigma_{910})/\sqrt{8} \\
\boldsymbol{\Gamma}^2_{0\downarrow} & = & (\Sigma_{98}+\Sigma_{710}+i\,\Sigma_{79}+i\,\Sigma_{810})/4
\end{array}
\tag{9.14}
$$

9.2 Features of the Φ^{15}_{00}

(i) If we spontaneously break $SO(10)$ by giving a vev to Φ^{15}_{00}, the X_α bosons and all the $(2,2,6)$ bosons get mass.

(ii) The Φ^{15}_{00} singlet of $(1,1,15)$ can not be used in the Yukawa sector. It fails to produce Dirac mass terms for fermions.

(iii) In contrast to the Higgs scalar that spontaneously break the electroweak gauge symmetry, the Φ_{15} Higgs scalar has no quantum numbers. It has no electric charge, neither left- nor right-isospin, it has no $B-L$ number and no spin etc. It can acquire only $mass$. There by it has a very classical nature.

10. THE HIGGS MULTIPLETS: THE (16)-REPRESENTATION

10.1 The Structure

We had previously decomposed the spinors Ψ_L and Ψ_R with respect to $SU(3)_C \times U(1)_Q$ and also with respect to $SU(3)_C \times SU(2)_L \times U(1)_Y$ in § 4.3.1. It was pointed out there that the existence of electrically neutral and colorless fermion singlets make the 16's i.e., Ψ_L and Ψ_R eligible for the Higgs mechanism. These electrically neutral and colorless fermion singlets are indeed neutrino states given in eq. (4.19) and (4.20). In the following, we concentrate on using the 16 for representing Higgs fields. The approach is to assign complex scalar fields to the Ψ_L and Ψ_R spinors. Since the Higgs scalar have no spin, the subscripts L and R which denote chirality of the spinors should be interpreted differently. Let us use $+$ and $-$ instead of L and R respectively. Also we use the letters ϕ_+ and ϕ_- instead of Ψ_+ and the Ψ_-. As a matter of fact the ϕ_+ and the ϕ_- Higgs representation are understood to transform under the Σ_+ and Σ_- representations respectively. These are projections of the complete $SO(10)$ representation :

$$\Sigma_\pm = \frac{1}{2}\left(1 \pm \Gamma_{five}\right)\,\Sigma \quad , \qquad \phi_\pm = \frac{1}{2}\left(1 \pm \Gamma_{five}\right)\,\phi \tag{10.1}$$

If Γ_{five} in eq. (2.15) for a particular basis assumes the special form in eq. (2.16), then the complete $SO(10)$ representation and ϕ can be expressed as

$$\phi = \begin{bmatrix} \phi_+ \\ \phi_- \end{bmatrix}, \; \Sigma = \begin{bmatrix} \Sigma_+ & 0 \\ 0 & \Sigma_- \end{bmatrix} \tag{10.2}$$

This actually happens for basis C and B. Let us assign 32 scalar field ϕ_i with $i = (1, \ldots, 32)$ to the 32 dimensional ϕ. Then we obtain

$$\phi_+ = \begin{bmatrix} \vartheta_r^{2/3} \\ \vartheta_g^{2/3} \\ \vartheta_b^{2/3} \\ \vartheta^0 \\ \vartheta_r^{-1/3} \\ \vartheta_g^{-1/3} \\ \vartheta_b^{-1/3} \\ \vartheta^1 \\ \varphi_{\bar r}^{1/3} \\ \varphi_{\bar g}^{1/3} \\ \varphi_{\bar b}^{1/3} \\ \varphi^{-1} \\ -\varphi_{\bar r}^{-2/3} \\ -\varphi_{\bar g}^{-2/3} \\ -\varphi_{\bar b}^{-2/3} \\ -\varphi^0 \end{bmatrix}, \qquad \phi_- = \begin{bmatrix} \varphi_r^{2/3} \\ \varphi_g^{2/3} \\ \varphi_b^{2/3} \\ \bar\varphi^0 \\ \varphi_r^{-1/3} \\ \varphi_g^{-1/3} \\ \varphi_b^{-1/3} \\ \varphi^1 \\ \vartheta_{\bar r}^{1/3} \\ \vartheta_{\bar g}^{1/3} \\ \vartheta_{\bar b}^{1/3} \\ \vartheta^{-1} \\ -\vartheta_{\bar r}^{-2/3} \\ -\vartheta_{\bar g}^{-2/3} \\ -\vartheta_{\bar b}^{-2/3} \\ -\bar\vartheta^0 \end{bmatrix} \tag{10.3}$$

Here the complex scalar fields are explicitly defined as

$$\begin{aligned} \vartheta_r^{2/3} &= (\phi_1 + i\,\phi_2)/\sqrt{2} & \vartheta_r^{-1/3} &= (\phi_9 + i\,\phi_{10})/\sqrt{2} \\ \vartheta_g^{2/3} &= (\phi_3 + i\,\phi_4)/\sqrt{2} & \vartheta_g^{-1/3} &= (\phi_{11} + i\,\phi_{12})/\sqrt{2} \\ \vartheta_b^{2/3} &= (\phi_5 + i\,\phi_6)/\sqrt{2} & \vartheta_b^{-1/3} &= (\phi_{13} + i\,\phi_{14})/\sqrt{2} \\ \vartheta^0 &= (\phi_7 + i\,\phi_8)/\sqrt{2} & \vartheta^1 &= (\phi_{15} + i\,\phi_{16})/\sqrt{2} \end{aligned} \tag{10.4}$$

$$
\begin{aligned}
\varphi_{\bar r}^{1/3} &= (\phi_{17} + i\,\phi_{18})/\sqrt{2} & -\varphi_{\bar r}^{-2/3} &= (\phi_{25} + i\,\phi_{26})/\sqrt{2} \\
\varphi_{\bar g}^{1/3} &= (\phi_{19} + i\,\phi_{20})/\sqrt{2} & -\varphi_{\bar g}^{-2/3} &= (\phi_{27} + i\,\phi_{28})/\sqrt{2} \\
\varphi_{\bar b}^{1/3} &= (\phi_{21} + i\,\phi_{22})/\sqrt{2} & -\varphi_{\bar b}^{-2/3} &= (\phi_{29} + i\,\phi_{30})/\sqrt{2} \\
\varphi^{-1} &= (\phi_{23} + i\,\phi_{24})/\sqrt{2} & -\varphi^{0} &= (\phi_{31} + i\,\phi_{32})/\sqrt{2}
\end{aligned}
\tag{10.5}
$$

Note that ϕ_+ contains the above 16 fields, ϕ_- contains the charge conjugated fields. i.e., $\overline{16}$. The various charges carried by these 32 fields are shown in the superscripts as well as in the subscripts. The superscripts denote electric-charges, subscripts denote $SU(3)$-color. The isospin charges are suppressed. However the fields denoted with φ carry only right isospin and the fields denoted with ϑ carry only left isospin. Note that all these Higgs fields carry $B-L$ numbers. The fields in ϕ_- appear in a different order than those in ϕ_+. Actually they are related over the C and P transformations. We have

$$
P\,\phi = \phi_P = P \begin{bmatrix} \phi_+ \\ \phi_- \end{bmatrix} = \begin{bmatrix} \phi_- \\ -\phi_+ \end{bmatrix} \tag{10.6}
$$

Here ϕ_P should transform under the P transformed representation given in eq. (4.24). Under charge conjugation we have

$$
C\,\phi = \phi_C = C \begin{bmatrix} \phi_+ \\ \phi_- \end{bmatrix} = \begin{bmatrix} A\,\phi_- \\ -A\,\phi_+ \end{bmatrix} \tag{10.7}
$$

where A is given in eq.(4.23) and ϕ_C should transform under the conjugated representation given in eq.(4.21). Furthermore the decomposition of the 32 dimensional Higgs multiplet is given in § 10.7.

10.2 The Primary Descent

Looking at the decomposition of the 32 dimensional Higgs multiplet in § 10.7, we see that there are no singlet fields in the second row. Therefore the $SU(4) \times SU(2)_L \times SU(2)_R$ symmetry can not be spontaneously broken down to $SU(3) \times SU(2)_L \times SU(2)_R \times U(1)_{B-L}$ via a Higgs scalar in the 32. In the third row there are the $(1,1)_0$ and $(\bar 1,1)_0$ singlet Higgs fields. These singlets correspond to φ^0 and its charge conjugate $\bar\varphi^0$ respectively. They carry zero hypercharge and are $SU(3)_c \times U(1)_Q$ singlets. These can be employed in the following two descents

$$
\begin{array}{lrcl}
(i) & SU(3)_c \times SU(2)_L \times SU(2)_R \times U(1)_{B-L} & \to & SU(3)_c \times SU(2)_L \times U(1)_Y \\
(ii) & SU(3)_c \times SU(2)_R \times U(1)_{Y'} & \to & SU(3)_c \times U(1)_Q
\end{array}
$$

where the first one will be called the *primary descent*. Because it spontaneously breaks the left-right symmetric symmetry down to the electroweak gauge symmetry. If all scalar fields are set to zero and φ^0 is given a vev, the resulting ϕ^{vac} expression will have two non zero entries. We have

$$
\phi_+^{vac} = \begin{bmatrix} 0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\ -\langle\varphi^0\rangle \end{bmatrix} \quad , \quad \phi_-^{vac} = \begin{bmatrix} 0\\0\\0\\ \langle\varphi^0\rangle^* \\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0 \end{bmatrix} \quad , \quad \phi_{vac} = \begin{bmatrix} \phi_+^{vac} \\ \phi_-^{vac} \end{bmatrix} \tag{10.8}
$$

Let us define $\langle\varphi^0\rangle = v_R \exp(-i\,\theta_R)$. We will refer to these two multiplets as $16'_+$ and $16'_-$ respectively. The residual symmetries under which the vacuum is left invariant can be found through infinitesimal transformations.

These are manifest through the following set of operations.

$$
\begin{array}{lll}
L_1\,\phi_{vac} & = 0 \\
L_2\,\phi_{vac} & = 0 \\
L_3\,\phi_{vac} & = 0 \\
R_1\,\phi_{vac} & \neq 0 \\
R_2\,\phi_{vac} & \neq 0 \\
R_3\,\phi_{vac} & \neq 0
\end{array}
\qquad
\begin{array}{rll}
S_i\,\phi_{vac} & = 0 & i=(0,\ldots,12) \\
S_i\,\phi_{vac} & \neq 0 & i=(13,\ldots,24) \\
U_i\,\phi_{vac} & = 0 & i=(1,\ldots,8) \\
U_i\,\phi_{vac} & \neq 0 & i=(9,\ldots,15) \\
(R_3+(B-L)/2)\;\phi_{vac} & = (Y/2)\;\phi_{vac}=0 & \\
(L_3+Y/2)\;\phi_{vac} & = \;Q\;\phi_{vac}=0 &
\end{array}
\tag{10.9}
$$

Since the amount of breaking in R_3 and U_{B-L} are opposite in sign and equal in strength, the hypercharge remains a residual symmetry of the vacuum.

10.3 The Secondary Descent

Looking at the decomposition of the 32 dimensional Higgs multiplet in § 10.7, we see that each of the $(1,2)_{-1}$ and $(\bar{1},2)_1$ multiplets in the second line decompose under $SU(3)\times U(1)_Q$ into singlets. These colorless and electrically neutral fields are ϑ^0 and its charge conjugate $\bar{\vartheta}^0$ respectively. These can be employed in the following two descents. We have

$$
\begin{array}{rrcl}
(i) & SU(3)_c\times SU(2)_L\times SU(2)_R\times U(1)_{B-L} & \rightarrow & SU(3)_c\times SU(2)_R\times U(1)_{Y'} \\
(ii) & SU(3)_c\times SU(2)_L\times U(1)_Y & \rightarrow & SU(3)_c\times U(1)_Q
\end{array}
$$

The second one will be called the *secondary descent*. It is complementary to the primary descent. They can be used together. Meanwhile the secondary descent above is formally similar to the spontaneous breakdown implemented in the electroweak theory where the Higgs fields is assigned to a spinor, strictly speaking a doublet. If all fields are set to zero and ϑ is assigned a vev, we obtain

$$
\phi_+^{vac}=\begin{bmatrix}0\\0\\0\\\langle\vartheta^0\rangle\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\end{bmatrix}\;,\qquad
\phi_-^{vac}=\begin{bmatrix}0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\-\langle\vartheta^0\rangle^*\end{bmatrix}\;,\qquad
\phi_{vac}=\begin{bmatrix}\phi_+^{vac}\\\phi_-^{vac}\end{bmatrix}
\tag{10.10}
$$

Let us define $\langle\vartheta^0\rangle=v_L\exp(-i\,\theta_L)$. We will refer to these two multiplets as $16''_+$ and $16''_-$. The residual symmetries under which the vacuum is left invariant can be obtained through the following operations

$$
\begin{array}{ll}
L_1\,\phi_{vac} & \neq 0 \\
L_2\,\phi_{vac} & \neq 0 \\
L_3\,\phi_{vac} & \neq 0 \\
R_1\,\phi_{vac} & = 0 \\
R_2\,\phi_{vac} & = 0 \\
R_3\,\phi_{vac} & = 0
\end{array}
\qquad
\begin{array}{rll}
S_i\,\phi_{vac} & = 0 & i=(4,5,6,10,11,12,16,17,18,22,23,24) \\
S_i\,\phi_{vac} & \neq 0 & i=(1,2,3,7,8,9,13,14,15,19,20,21) \\
U_i\,\phi_{vac} & = 0 & i=(1,\ldots,8) \\
U_i\,\phi_{vac} & \neq 0 & i=(9,\ldots,15) \\
Y\,\phi_{vac} & \neq 0 & \\
(L_3+Y/2)\,\phi_{vac} & = \;Q\;\phi_{vac}=0 &
\end{array}
\tag{10.11}
$$

In the last expression it is seen that the amount of breaking in L_3 and in Y add up to zero so that $U(1)_Q$ remains a residual symmetry of the vacuum.

10.4 Left-Right Asymmetry

We know that the $SU(3)\times U(1)_Q$ gauge symmetry is preceded by the electroweak gauge symmetry as we approach the Fermi mass scale. Therefore the primary descent should take place at some energy scale which lies above the mass scale of the secondary descent provided that we restrict ourself to use both $\langle\varphi^0\rangle$ and $\langle\vartheta^0\rangle$ in a spontaneous symmetry breakdown. We should have $\langle\varphi^0\rangle \;>\; \langle\vartheta^0\rangle$. This hierarchy gives rise to the following pattern:

$$SU(3)\times SU(2)_L\times SU(2)_R\times U(1)_{B-L} \xrightarrow{\langle\varphi^0\rangle} SU(3)\times SU(2)_L\times U(1)_Y \xrightarrow{\langle\vartheta^0\rangle} SU(3)\times U(1)_Q \tag{10.12}$$

If we reverse this inequality such that $\langle\varphi^0\rangle \;<\; \langle\vartheta^0\rangle$ then things change and we encounter a different pattern:

$$SU(3)\times SU(2)_L\times SU(2)_R\times U(1)_{B-L} \xrightarrow{\langle\vartheta^0\rangle} SU(3)\times SU(2)_R\times U(1)_{Y'} \xrightarrow{\langle\varphi^0\rangle} SU(3)\times U(1)_Q \tag{10.13}$$

This latter pattern is symmetric to the former under parity transformation. According to the above descents where the initial symmetry is common, a left-right symmetric intermediate vacuum could have been possible if the initial symmetry were not spontaneously broken at all, or it could have been possible if the initial symmetry were spontaneously broken with equal strengths of the two vevs such that $\langle\varphi^0\rangle = \langle\vartheta^0\rangle$. In both cases we would not observe an intermediate electroweak gauge symmetry at all. The current phenomenology restricts us to the condition that $\langle\varphi^0\rangle \;<\; \langle\vartheta^0\rangle$. The above situation is not special to the patterns that are brought about by the two 16's. It also occurs when we consider the $(1,3,10)\oplus(3,1,\bar{1}0)$ Higgs fields in § 7.1. As a result, in the realm of $SO(10)$ one *should* consider a primary descent which is considerably elevated with respect to the secondary descent.

10.5 Features of the $16'$ and $16''$

Here we shortly consider how the $16'$ in eq. (10.8) and the $16''$ in eq. (10.10) behave in the Higgs mechanism and in the Yukawa sector.

(i) The $16'$ gives mass to the A_α, A'_α, X_α, W_R^+, W_R^0, W_R^-, X_{B-L} gauge fields. The $16''$ gives mass to the Y_α, A_α, X_α, W_L^+, W_L^0, W_L^-, X_{B-L} gauge fields.

(ii) Since the $16'$ and $16''$ are no square matrices, the possible mass terms that can be constructed for fermions, have mass dimensions greater than 4. This spoils renormalisibility. Dimension 5 terms which produce Dirac masses for all fermions can be obtained through terms like

$$\Big((16'_\pm)^\dagger\,(16''_\mp)\Big)\,(\overline{\Psi}_L\Psi_R) \to \big(\langle\varphi^0\rangle^*\langle\vartheta^0\rangle^* + \langle\varphi^0\rangle\langle\vartheta^0\rangle\big)\,\overline{f_L}\,f_R \tag{10.14}$$

where either the $(+-)$ or $(-+)$ pairs should be considered. These vertices are shown in Fig. (10.1). At the vertices two Higgs scalars carry the correct quantum charges away. Other Dirac mass terms only for neutrinos can obtained through terms like

$$\overline{\Big((16''_\pm)^\dagger\,(\Psi_L)\Big)}\,\Big((16'_\mp)^\dagger\,(\Psi_R)\Big) \to \langle\varphi^0\rangle^*\langle\vartheta^0\rangle^*\,\overline{\nu_L}\,\nu_R + \langle\varphi^0\rangle\langle\vartheta^0\rangle\,\overline{(\nu_L)^c}\,(\nu_R)^c \tag{10.15}$$

Majorana mass terms for neutrinos are also possible with the following two terms

$$\overline{\Big((16'_+)^\dagger\,(\Psi_L)\Big)}\,\Big((16'_-)^\dagger\,(\Psi_R)\Big) \to \langle\varphi^0\rangle\langle\varphi^0\rangle\;\;\overline{(\nu_R)^c}\,\nu_R \tag{10.16}$$

$$\overline{\Big((16''_+)^\dagger\,(\Psi_L)\Big)}\,\Big((16''_-)^\dagger\,(\Psi_R)\Big) \to \langle\vartheta^0\rangle\langle\vartheta^0\rangle\;\;\overline{\nu_L}\,(\nu_L)^c \tag{10.17}$$

Note that the primary descent produces Majorana mass terms for ν_R. The secondary descent produces Majorana masses for ν_L. These vertices are also shown in Fig. (10.1).

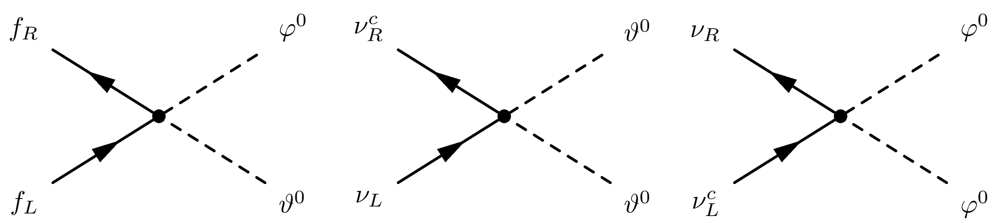

Fig. 10.1: Mass terms for fermions produced via Higgs scalars in the 16's. The first one produces Dirac mass terms for all fermions and the latter two produce Majorana mass terms for neutrinos.

10.6 Lepton and Quark Masses via the 16

The neutrinos may have simultaneously Dirac and Majorana masses. The charged leptons and quarks can have only dirac masses. Using the Higgs multiplets $16'_\pm$ and $16''_\pm$, which were defined in § 10.2 and § 10.3 respectively, we can rewrite the above dimension five mass terms for neutrinos and as well as for charged leptons and quarks in a nicer way. Let us introduce two Higgs fields ϕ_1 and ϕ_2. We have

$$\phi_1 = \begin{pmatrix} 16'_+ \\ 16'_- \end{pmatrix} \qquad \phi_2 = \begin{pmatrix} 16''_+ \\ 16''_- \end{pmatrix} \tag{10.18}$$

These Higgs fields can be used to generate the mass terms in eqs. (10.14) to (10.17). The appropriate Yukawa Lagrangian reads

$$\begin{aligned} \mathbf{L}_Y = \frac{Y^D_{ij}}{2} &\left\{ \left(\overline{\phi}_1\, \phi_2\right) \left(\overline{\Psi}_i\, \Psi_j\right) + \left(\overline{\phi}_2\, \phi_1\right) \left(\overline{\Psi}_i\, \Psi_j\right) + \overline{\left(\phi_1^\dagger\, \Psi_i\right)} \left(\phi_2^\dagger\; \Psi_j\right) + \overline{\left(\phi_2^\dagger\, \Psi_i\right)} \left(\phi_1^\dagger\; \Psi_j\right) \right\} \\ &+ \left\{ Y^R_{ij} \overline{\left(\phi_1^\dagger\, \Psi_i\right)} \left(\phi_1^\dagger\; \Psi_j\right) + Y^L_{ij} \overline{\left(\phi_2^\dagger\, \Psi_i\right)} \left(\phi_2^\dagger\; \Psi_j\right) \right\} \end{aligned} \tag{10.19}$$

where $\overline{\phi_k} = \phi_k^\dagger P$ for $k = 1, 2$. In our representation P is like γ_0 in the Dirac-Pauli representation. Y^D_{ij}, Y^L_{ij} and Y^R_{ij} are Yukawa couplings. The summation is done over $i, j = (1, 2, 3)$, which denotes the family space. The first two terms in $\mathbf{L}_Y$ generate Dirac masses not only for neutrinos but also for the remaining fermions. It is useful to define the following neutrino fields

$$f_i = \frac{e^{-i\,\theta_L}\nu^i_L - e^{i\,\theta_L}(\nu^i_L)^c}{\sqrt{2}} = \frac{\langle\phi_2^\dagger\rangle\Psi_i}{v_L}, \qquad \bar{f}_i = \frac{e^{i\,\theta_L}\overline{\nu^i_L} - e^{-i\,\theta_L}\overline{(\nu^i_L)^c}}{\sqrt{2}} = \frac{\overline{\langle\phi_2^\dagger\rangle\Psi_i}}{v_L}$$

$$F_i = \frac{e^{i\,\theta_R}\nu^i_R - e^{-i\,\theta_R}(\nu^i_R)^c}{\sqrt{2}} = \frac{\langle\phi_1^\dagger\rangle\Psi_i}{v_R}, \qquad \bar{F}_i = \frac{e^{-i\,\theta_R}\overline{\nu^i_R} - e^{i\,\theta_R}\overline{(\nu^i_R)^c}}{\sqrt{2}} = \frac{\overline{\langle\phi_1^\dagger\rangle\Psi_i}}{v_R}$$

Here F_i and f_i carry flavor index $i = (1, 2, 3)$ running over all neutrino species. Theses neutrino fields will serve as an eigenbasis for writing the neutrino mass matrix. The eigenbasis has the property that $F = F^c$ and $f = f^c$. The phase $\exp i\,\theta_L$ and $\exp i\,\theta_R$ in f and F are originating from the vevs. The Dirac and Majorana mass terms for neutrinos resulting from L_Y can be collected into a separate Lagrangian $\mathbf{L}^\nu_Y$ through a mass matrix by means of the formerly defined quantities F and f. The terms in the upper line in $\mathbf{L}_Y$ can be brought to the same form with the first term in the lower line if we set $\theta_L + \theta_R = \pi$. This simplifies the Lagrangian. We get

$$\mathbf{L}^\nu_Y = \begin{bmatrix} \bar{f}_i & \bar{F}_i \end{bmatrix} \begin{bmatrix} M^L_{ij} & M^D_{ij} \\ M^D_{ij} & M^R_{ij} \end{bmatrix} \begin{bmatrix} f_j \\ F_j \end{bmatrix} = \begin{bmatrix} \bar{f}_i & \bar{F}_i \end{bmatrix} \mathcal{M} \begin{bmatrix} f_j \\ F_j \end{bmatrix} \tag{10.20}$$

Here M^D_{ij} , M^L_{ij} and M^R_{ij} are 3×3 matrices in the flavor space and are explicitly defined as

$$M^D_{ij} = Y^D_{ij}\;\; v_L\, v_R\,, \quad M^L_{ij} = Y^L_{ij}\;\; v_L^2\,, \quad M^R_{ij} = Y^R_{ij}\;\; v_R^2 \tag{10.21}$$

Diagonalizing the mass matrix yields the masses of ν_L and ν_R. Without adapting any particular texture for the Yukawa couplings, the neutrino masses are uniformly predicted as

$$\begin{aligned} m^2_{\nu_L} &= 0 \quad \text{, Exactly zero} \\ m^2_{\nu_R} &= v_L^2 + v_R^2 \end{aligned} \tag{10.22}$$

where we have suppressed over flavor indices and assumed the matrices M^D_{ij}, M^L_{ij} and M^R_{ij} to be scalar entries. The couplings are chosen to satisfy the condition $Y^R_{ij} \equiv Y^L_{ij} \equiv Y^D_{ij} \equiv 1$. The values for $\langle\vartheta\rangle$ and $\langle\varphi\rangle$ depend on the model.

10.7 Decomposition of the $16 \oplus \overline{16}$

We proceed with the decomposition of the 32 complex scalar fields under $SU(4) \times SU(2)_L \times SU(2)_R$, $SU(3) \times SU(2)_L \times SU(2)_R \times U(1)_{B-L}$ and $SU(3) \times SU(2)_L \times U(1)_Y$ respectively. We have

$$
\begin{array}{lllllll}
32 & = & (4,2,1) & + & (\bar{4},1,2) & + & (4,1,2) & + & (\bar{4},2,1) & & & \\
32 & = & (3,2,1)_{+1/3} & + & (1,2,1)_{-1} & + & (\bar{3},1,2)_{-1/3} & + & (\bar{1},1,2)_{1} & + & (3,1,2)_{+1/3} & + \\
 & & (1,1,2)_{-1} & + & (\bar{3},2,1)_{-1/3} & + & (\bar{1},2,1)_{1} & & & & & \\
32 & = & (3,2)_{1/3} & + & (1,2)_{-1} & + & (\bar{3},1)_{2/3} & + & (\bar{3},1)_{-4/3} & + & (\bar{1},1)_{2} & + \\
 & & (\bar{1},1)_{0} & + & (3,1)_{4/3} & + & (3,1)_{-2/3} & + & (1,1)_{0} & + & (1,1)_{-2} & + \\
 & & (\bar{3},2)_{-1/3} & + & (\bar{1},2)_{1} & & & & & & &
\end{array}
\tag{10.23}
$$

11. AN $SO(10)$ MODEL

11.1 Its Higgs Lagrangian

In this section, we will construct a suitable Higgs Lagrangian whose potential part should consist of those Higgs multiplets which are physically *most relevant*. These Higgs multiplets are summarized in Table (11.1). The same multiplets are also shown in Fig. (11.1) where we have clearly demonstrated how these multiplets relate certain descents in the Higgs mechanism.

Why do we need especially these multiplets ? And why are some other multiplets excluded ? Let us try to answer these questions: (i) Naturally we expect that the $SO(10)$ or the $SU(4) \times SU(2)_L \times SU(2)_R$ gauge symmetry can only exist at some *extremely* high energy scale, high in comparison to the Fermi scale and the known Fermion masses. If we were to break solely the $SO(10)$ or the $SU(4) \times SU(2)_L \times SU(2)_R$ gauge symmetry with one of the multiplets in the 10 or in the 126 with some vacuum expectation value at the order of this *extremely* high scale then the fermions would also receive masses at this order. This is obviously in contradiction with our observations of the known fermion masses. (ii) The gauge bosons which lie in the coset of $SO(10)$ and $SU(4) \times SU(2)_L \times SU(2)_R$ should be *extremely massive*, otherwise this would again be in contradiction with the observation of the extremely long life time of the proton. Therefore from (i) and (ii) it is easy to conclude that we need some multiplet which can endow the $A_\alpha, A'_\alpha, Y_\alpha, Y'_\alpha$ and X_α gauge bosons with *masses* but leaves simultaneously the fermions *massless*. This is easily met by the inclusion of the $(1,1,15)$ of the 45 into the Higgs potential. Note that the $(1,1,1)$ singlet of the 210 can also meet the same requirement with the *exception* that it leaves the X_α bosons *massless*. Nevertheless the inclusion of the $(1,1,1)$ singlet of the 210 in the Higgs mechanism might not be dangerous. Remember that the X_α's can not mediate nucleon decays alone. Therefore we will also include the $(1,1,1)$ singlet of the 210 into the Higgs potential, in the hope that the inclusion of intermediate $SU(4) \times SU(2)_L \times SU(2)_R$ gauge symmetry in our model provides a better description of nature than its absence. We will come to this point again.

The Yukawa sector and the Higgs sector are of course not independent, because the scalars which spontaneously break the symmetry of the vacuum can also couple to fermions and endow them with some masses. The multiplets that can be used to account for fermion masses are in the 10 and in the 126 Higgs representations. We will make full use of these scalars, allowing also Majorana masses for neutrinos via the $(1,3,10)$ and $(3,1,\bar{10})$ scalars. Primarily there is nothing against this, and secondarily it is also reasonable due to the following facts: (iii) The right-handed neutrino is missing and if it exists at all, it should have some mass *above* the Fermi scale. (iv) The left-handed neutrinos, unlike the *charged leptons* and *quarks*, are almost massless, with masses extremely *below* the Fermi scale. (v) Remember that the $SO(10)$ Higgs sector should also be able to account for the left-right asymmetry which was discussed before; We expect that the vacuum expectation value of $(1,3,10)$ should be considerably elevated with respect to that of $(3,1,\bar{10})$. The last three steps (iii), (iv) and (v) are intrinsically tied together. The nature of the neutrino masses as well as the left-right asymmetry are better understood if one lets the neutrinos acquire both Dirac and Majorana masses. This leads to mass matrices which produce neutrino masses similar to that given in eq. (10.22).

<table>
<tr><th colspan="8">The Most Relevant Higgs Multiplets</th></tr>
<tr><td>210</td><td>45</td><td colspan="4">126</td><td colspan="2">10</td></tr>
<tr><td>$(1,1,1)$</td><td>$(1,1,15)$</td><td>$(3,1,\bar{10})$</td><td>$(1,3,10)$</td><td colspan="2">$(2,2,15)$</td><td colspan="2">$(2,2,1)$</td></tr>
<tr><td>Φ^s_{00}</td><td>Φ^{15}_{00}
+ 36 Goldstone modes</td><td>$\Phi^{10}_{\downarrow 0}$</td><td>$\Phi^{10}_{0\uparrow}$</td><td>$\Phi^{15}_{\uparrow\downarrow}$</td><td>$\Phi^{15}_{\downarrow\uparrow}$</td><td>$\Phi^{5}_{\uparrow\downarrow}$</td><td>$\Phi^{5}_{\downarrow\uparrow}$</td></tr>
</table>

Tab. 11.1: The physically most relevant Higgs multiplets in $SO(10)$

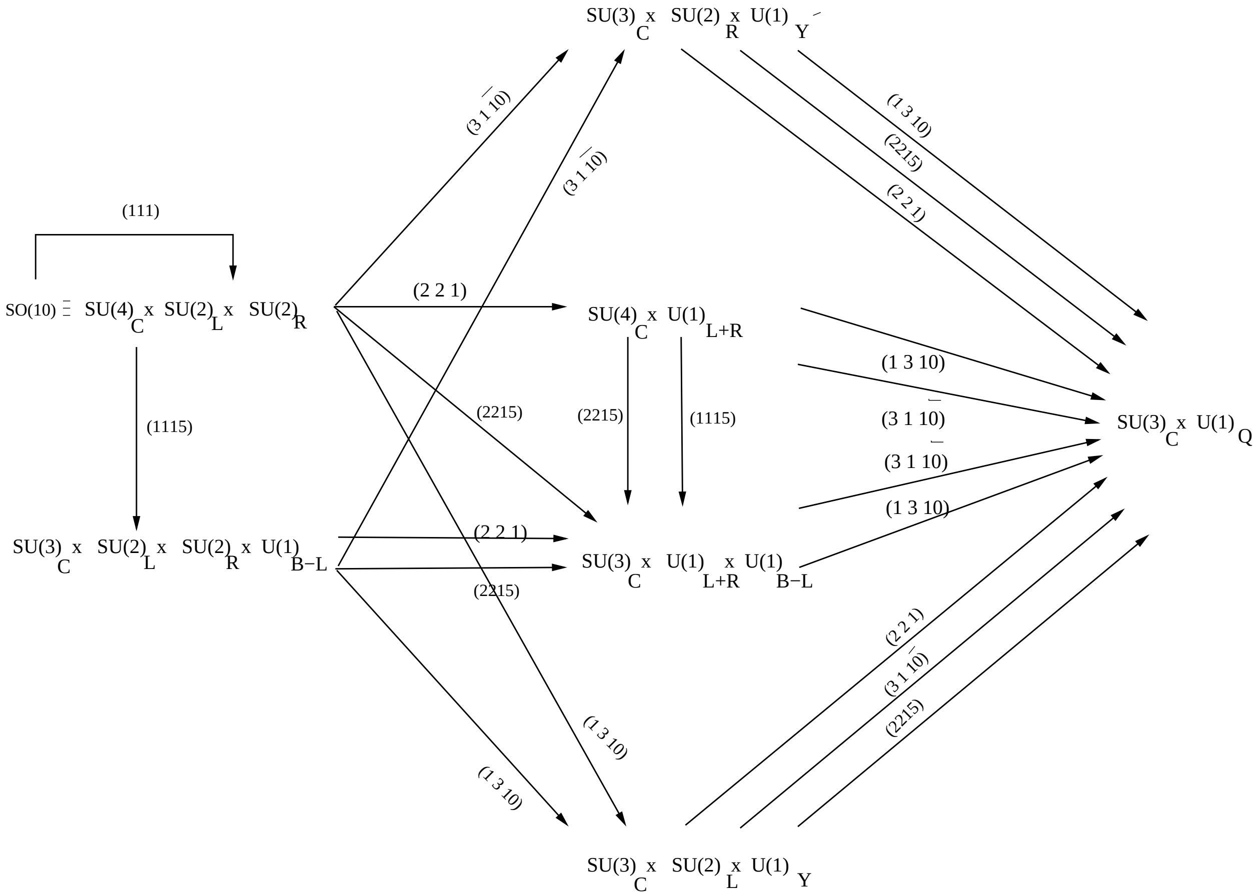

Fig. 11.1: An illustration showing how the Higgs fields break the initial and intermediate symmetries.

Another problem one has to account for are the degrees of freedom in the gauge sector. As seen from Table (11.1), we have introduced totally 36 Goldstone bosons. These are chosen out of the Higgs scalars of the 45 and will receive no vacuum expectation value. In the spontaneous breakdown of the $SO(10)$ symmetry down to $SU(3)_c \times U(1)_Q$ there exist exactly $45 - 8 - 1 = 36$ gauge fields which become massive. It could be thought that these 36 gauge fields eat those 36 Goldstone bosons and gain an extra longitudinal degree of freedom. The initial *symmetry* of the vacuum even after spontaneous symmetry breakdown does *not* really get lost [70]. The 36 Goldstone bosons are chosen out of the 45 because at the minimum, it should be possible to parameterize the 36 Goldstone fields via a unitary gauge as done in the electroweak theory. This is physically a consistent way of gauging away the unwanted Goldstone bosons [51][71].

In the following, we introduce some shorthand definitions for the Higgs fields in Table (11.1) which simplify the expressions entering the Higgs Lagrangian. We have

$$\begin{aligned}
\Phi_{111} =\ & \Phi^s_{00}\,\mathbf{\Gamma}^s_{00}\ , \\
\Phi_{45} =\ & (1/2)\,(\Phi^1_{\uparrow\uparrow}\,\mathbf{\Gamma}^1_{\uparrow\uparrow} + \Phi^2_{\uparrow\uparrow}\,\mathbf{\Gamma}^2_{\uparrow\uparrow} + \Phi^3_{\uparrow\uparrow}\,\mathbf{\Gamma}^3_{\uparrow\uparrow} + \Phi^4_{\uparrow\downarrow}\,\mathbf{\Gamma}^4_{\uparrow\downarrow} + \Phi^5_{\uparrow\downarrow}\,\mathbf{\Gamma}^5_{\uparrow\downarrow} + \Phi^6_{\uparrow\downarrow}\,\mathbf{\Gamma}^6_{\downarrow\downarrow} + \\
& \Phi^7_{\downarrow\downarrow}\,\mathbf{\Gamma}^7_{\downarrow\downarrow} + \Phi^8_{\downarrow\downarrow}\,\mathbf{\Gamma}^8_{\downarrow\downarrow} + \Phi^9_{\downarrow\downarrow}\,\mathbf{\Gamma}^9_{\downarrow\downarrow} + \Phi^{10}_{\downarrow\uparrow}\,\mathbf{\Gamma}^{10}_{\downarrow\uparrow} + \Phi^{11}_{\downarrow\uparrow}\,\mathbf{\Gamma}^{11}_{\downarrow\uparrow} + \Phi^{12}_{\downarrow\uparrow}\,\mathbf{\Gamma}^{12}_{\downarrow\uparrow} + \\
& \Phi^{13}_{\downarrow\uparrow}\,\mathbf{\Gamma}^{13}_{\downarrow\uparrow} + \Phi^{14}_{\downarrow\uparrow}\,\mathbf{\Gamma}^{14}_{\downarrow\uparrow} + \Phi^{15}_{\downarrow\uparrow}\,\mathbf{\Gamma}^{15}_{\downarrow\uparrow} + \Phi^{16}_{\downarrow\downarrow}\,\mathbf{\Gamma}^{16}_{\downarrow\downarrow} + \Phi^{17}_{\downarrow\downarrow}\,\mathbf{\Gamma}^{17}_{\downarrow\downarrow} + \Phi^{18}_{\downarrow\downarrow}\,\mathbf{\Gamma}^{18}_{\downarrow\downarrow} + \\
& \Phi^{19}_{\uparrow\downarrow}\,\mathbf{\Gamma}^{19}_{\uparrow\downarrow} + \Phi^{20}_{\uparrow\downarrow}\,\mathbf{\Gamma}^{20}_{\uparrow\downarrow} + \Phi^{21}_{\uparrow\downarrow}\,\mathbf{\Gamma}^{21}_{\uparrow\uparrow} + \Phi^{22}_{\uparrow\uparrow}\,\mathbf{\Gamma}^{22}_{\uparrow\downarrow} + \Phi^{23}_{\uparrow\uparrow}\,\mathbf{\Gamma}^{23}_{\uparrow\uparrow} + \Phi^{24}_{\uparrow\uparrow}\,\mathbf{\Gamma}^{24}_{\uparrow\uparrow} + \\
& \Phi^9_{00}\,\mathbf{\Gamma}^9_{00} + \Phi^{10}_{00}\,\mathbf{\Gamma}^{10}_{00} + \Phi^{11}_{00}\,\mathbf{\Gamma}^{11}_{00} + \Phi^{12}_{00}\,\mathbf{\Gamma}^{12}_{00} + \Phi^{13}_{00}\,\mathbf{\Gamma}^{13}_{00} + \Phi^{14}_{00}\,\mathbf{\Gamma}^{14}_{00} + \\
& \Phi^{15}_{00}\,\mathbf{\Gamma}^{15}_{00} + \Phi^1_{\uparrow 0}\,\mathbf{\Gamma}^1_{\uparrow 0} + \Phi^3_{\hat{0}0}\,\mathbf{\Gamma}^3_{\hat{0}0} + \Phi^2_{\downarrow 0}\,\mathbf{\Gamma}^2_{\downarrow 0} + \Phi^1_{0\uparrow}\,\mathbf{\Gamma}^1_{0\uparrow} + \Phi^3_{0\hat{0}}\,\mathbf{\Gamma}^3_{0\hat{0}} + \\
& \Phi^2_{0\downarrow}\,\mathbf{\Gamma}^2_{0\downarrow}\ )\ , \qquad (11.1) \\
\Phi_{31\bar{10}} =\ & \left(\Phi^{10}_{\downarrow 0}\,\mathbf{\Gamma}^{10}_{\downarrow 0}\right)/32\ ,
\end{aligned}$$

$$\Phi_{1310} = \left(\Phi^{10}_{0\uparrow}\, \mathbf{\Gamma}^{10}_{0\uparrow}\right)/32\,,$$

$$\Phi_{2215} = \left(\Phi^{15}_{\uparrow\downarrow}\, \mathbf{\Gamma}^{15}_{\uparrow\downarrow} + \Phi^{15}_{\downarrow\uparrow}\, \mathbf{\Gamma}^{15}_{\downarrow\uparrow}\right)/\sqrt{384}\,, \tag{11.2}$$

$$\Phi_{221} = \left(\Phi^{5}_{\uparrow\downarrow}\, \mathbf{\Gamma}^{5}_{\uparrow\downarrow} + \Phi^{5}_{\downarrow\uparrow}\mathbf{\Gamma}^{5}_{\downarrow\uparrow}\right)/\sqrt{32}\,,$$

The properties of these Higgs fields were previously studied in detail in § 8, 9, 7 and 6 respectively. The vacuum expectation values of the above given Higgs fields will be shown with $\langle\ \ \rangle$. They are chosen as

$$\begin{aligned}
\langle\Phi_{111}\rangle &= \mathbf{z}\,\mathbf{\Gamma}^{s}_{00}\ , & \mathbf{z} &= z\ (\cos\varphi\ +\ i\ \sin\varphi\,) \\
\langle\Phi_{45}\rangle &= \mathbf{x}\,\mathbf{\Gamma}^{15}_{00}\ , & \mathbf{x} &= x\ (\cos\vartheta\ +\ i\ \sin\vartheta\,) \\
\langle\Phi_{31\bar{1}\bar{0}}\rangle &= \mathbf{v}_L\,\mathbf{\Gamma}^{10}_{\uparrow 0}\ , & \mathbf{v}_L &= v_L\ (\cos\beta\ +\ i\ \sin\beta\,) \\
\langle\Phi_{1310}\rangle &= \mathbf{v}_R\,\mathbf{\Gamma}^{10}_{0\uparrow}\ , & \mathbf{v}_R &= v_R\ (\cos\gamma\ +\ i\ \sin\gamma\,) \\
\langle\Phi_{2215}\rangle &= \mathbf{u}\,\mathbf{\Gamma}^{15}_{\uparrow\downarrow} + \mathbf{v}\,\mathbf{\Gamma}^{15}_{\downarrow\uparrow}\ , & \mathbf{u} &= u\ (\cos\delta\ +\ i\ \sin\delta\,)\ \ ,\quad \mathbf{v} = v\ (\cos\theta\ +\ i\ \sin\theta\,) \\
\langle\Phi_{221}\rangle &= \mathbf{k}\,\mathbf{\Gamma}^{5}_{\uparrow\downarrow} + \mathbf{k}^{\dagger}\,\mathbf{\Gamma}^{5}_{\downarrow\uparrow}\ , & \mathbf{k} &= k\ (\cos\alpha\ +\ i\ \sin\alpha\,)
\end{aligned} \tag{11.3}$$

where $\{\mathbf{z},\mathbf{x},\mathbf{v_L},\mathbf{v_R},\ \mathbf{u},\mathbf{v},\mathbf{k}\}$ are complex valued quantities. The phases assigned to these vevs are defined as in the right hand side above. Let us construct the following potential terms by using the multiplets defined in eqs. (11.1) and (11.2):

$$\begin{aligned}
\mathbf{U}_{111} &= -\mu_1 Tr\{\Phi_{111}\Phi^{\dagger}_{111}\} + 32\ \lambda_1 Tr\{(\Phi_{111}\Phi^{\dagger}_{111})^2\} + \beta_1(Tr\{\Phi_{111}\Phi^{\dagger}_{111}\})^2 \\
\mathbf{U}_{45} &= -\mu_5 Tr\{\Phi_{45}\Phi^{\dagger}_{45}\} + (96/7)\ \lambda_5 Tr\{(\Phi_{45}\Phi^{\dagger}_{45})^2\} + \beta_5(Tr\{\Phi_{45}\Phi^{\dagger}_{45}\})^2 \\
\mathbf{U}_{31\bar{1}\bar{0}} &= -\mu_4 Tr\{\Phi_{(31\bar{1}\bar{0})}\Phi^{\dagger}_{31\bar{1}\bar{0}}\} + \lambda_4 Tr\{(\Phi_{31\bar{1}\bar{0}}\Phi^{\dagger}_{31\bar{1}\bar{0}})^2\} + \beta_4(Tr\{\Phi_{31\bar{1}\bar{0}}\Phi^{\dagger}_{31\bar{1}\bar{0}}\})^2 \\
\mathbf{U}_{1310} &= -\mu_4 Tr\{\Phi_{1310}\Phi^{\dagger}_{1310}\} + \lambda_4 Tr\{(\Phi_{1310}\Phi^{\dagger}_{1310})^2\} + \beta_4(Tr\{\Phi_{1310}\Phi^{\dagger}_{1310}\})^2 \\
\mathbf{U}_{2215} &= -\mu_3 Tr\{\Phi_{2215}\Phi^{\dagger}_{2215}\} + (24/7)\ \lambda_3 Tr\{(\Phi_{2215}\Phi^{\dagger}_{2215})^2\} + \beta_3(Tr\{\Phi_{2215}\Phi^{\dagger}_{2215}\})^2 \\
\mathbf{U}_{221} &= -\mu_2 Tr\{\Phi_{221}\Phi^{\dagger}_{221}\} + 32\ \lambda_2 Tr\{(\Phi_{221}\Phi^{\dagger}_{221})^2\} + \beta_2(Tr\{\Phi_{221}\Phi^{\dagger}_{221}\})^2
\end{aligned} \tag{11.4}$$

The labelling might seem to be extravagant. But a more economical notation could make things less traceable. The funny numbers appearing in front of the coupling strengths are introduced to simplify the resulting expressions. They can be reabsorbed after one finishes solving for the minima. Let us continue with the potential terms that can be build by crossing the Higgs fields. These are

$$\begin{aligned}
\mathbf{U}^{45}_{111} &= 32\ \mu_{15} Tr\{\Phi_{45}\Phi^{\dagger}_{45}\Phi_{111}\Phi^{\dagger}_{111}\} + \lambda_{15} Tr\{\Phi_{45}\Phi^{\dagger}_{45}\} Tr\{\Phi_{111}\Phi^{\dagger}_{111}\} \\
\mathbf{U}^{31\bar{1}\bar{0}}_{111} &= 32\ \mu_{14} Tr\{\Phi_{31\bar{1}\bar{0}}\Phi^{\dagger}_{31\bar{1}\bar{0}}\Phi_{111}\Phi^{\dagger}_{111}\} + \lambda_{14} Tr\{\Phi_{31\bar{1}\bar{0}}\Phi^{\dagger}_{31\bar{1}\bar{0}}\} Tr\{\Phi_{111}\Phi^{\dagger}_{111}\} \\
\mathbf{U}^{1310}_{111} &= 32\ \mu_{14} Tr\{\Phi_{1310}\Phi^{\dagger}_{1310}\Phi_{111}\Phi^{\dagger}_{111}\} + \lambda_{14} Tr\{\Phi_{1310}\Phi^{\dagger}_{1310}\} Tr\{\Phi_{111}\Phi^{\dagger}_{111}\} \\
\mathbf{U}^{2215}_{111} &= 32\ \mu_{13} Tr\{\Phi_{2215}\Phi^{\dagger}_{2215}\Phi_{111}\Phi^{\dagger}_{111}\} + \lambda_{13} Tr\{\Phi_{2215}\Phi^{\dagger}_{2215}\} Tr\{\Phi_{111}\Phi^{\dagger}_{111}\}
\end{aligned} \tag{11.5}$$

$$\mathbf{U}_{111}^{221} = 32\,\mu_{12} Tr\{\Phi_{221}\Phi_{221}^{\dagger}\Phi_{111}\Phi_{111}^{\dagger}\} + \lambda_{12} Tr\{\Phi_{221}\Phi_{221}^{\dagger}\} Tr\{\Phi_{111}\Phi_{111}^{\dagger}\}$$

$$\mathbf{U}_{45}^{31\bar{1}\bar{0}} = (32/3)\,\mu_{54} Tr\{\Phi_{31\bar{1}\bar{0}}\Phi_{31\bar{1}\bar{0}}^{\dagger}\Phi_{45}\Phi_{45}^{\dagger}\} + \frac{\lambda_{54}}{2} Tr\{\Phi_{31\bar{1}\bar{0}}\Phi_{31\bar{1}\bar{0}}^{\dagger}\} Tr\{\Phi_{45}\Phi_{45}^{\dagger}\}$$

$$\mathbf{U}_{45}^{1310} = (32/3)\,\mu_{54} Tr\{\Phi_{1310}\Phi_{1310}^{\dagger}\Phi_{45}\Phi_{45}^{\dagger}\} + \lambda_{54} Tr\{\Phi_{1310}\Phi_{1310}^{\dagger}\} Tr\{\Phi_{45}\Phi_{45}^{\dagger}\}$$

$$\mathbf{U}_{45}^{2215} = (96/7)\,\mu_{53} Tr\{\Phi_{2215}\Phi_{2215}^{\dagger}\Phi_{45}\Phi_{45}^{\dagger}\} + \lambda_{53} Tr\{\Phi_{2215}\Phi_{2215}^{\dagger}\} Tr\{\Phi_{45}\Phi_{45}^{\dagger}\}$$

$$\mathbf{U}_{45}^{221} = 32\,\mu_{52} Tr\{\Phi_{221}\Phi_{221}^{\dagger}\Phi_{45}\Phi_{45}^{\dagger}\} + \lambda_{52} Tr\{\Phi_{221}\Phi_{221}^{\dagger}\} Tr\{\Phi_{45}\Phi_{45}^{\dagger}\}$$

$$\mathbf{U}_{31\bar{1}\bar{0}}^{1310} = \mu_{44} Tr\{\Phi_{1310}\Phi_{1310}^{\dagger}\} Tr\{\Phi_{31\bar{1}\bar{0}}\Phi_{31\bar{1}\bar{0}}^{\dagger}\} \tag{11.6}$$

$$\mathbf{U}_{31\bar{1}\bar{0}}^{2215} = (8/3)\,\mu_{43} Tr\{\Phi_{31\bar{1}\bar{0}}\Phi_{31\bar{1}\bar{0}}^{\dagger}\Phi_{2215}\Phi_{2215}^{\dagger}\} + \lambda_{43} Tr\{\Phi_{31\bar{1}\bar{0}}\Phi_{31\bar{1}\bar{0}}^{\dagger}\} Tr\{\Phi_{2215}\Phi_{2215}^{\dagger}\}$$

$$\mathbf{U}_{31\bar{1}\bar{0}}^{221} = (8/3)\,\mu_{42} Tr\{\Phi_{31\bar{1}\bar{0}}\Phi_{31\bar{1}\bar{0}}^{\dagger}\Phi_{221}\Phi_{221}^{\dagger}\} + \lambda_{42} Tr\{\Phi_{31\bar{1}\bar{0}}\Phi_{31\bar{1}\bar{0}}^{\dagger}\} Tr\{\Phi_{221}\Phi_{221}^{\dagger}\}$$

$$\mathbf{U}_{1310}^{2215} = (8/3)\,\mu_{43} Tr\{\Phi_{1310}\Phi_{1310}^{\dagger}\Phi_{2215}\Phi_{2215}^{\dagger}\} + \lambda_{43} Tr\{\Phi_{1310}\Phi_{1310}^{\dagger}\} Tr\{\Phi_{2215}\Phi_{2215}^{\dagger}\}$$

$$\mathbf{U}_{1310}^{221} = (8/3)\,\mu_{42} Tr\{\Phi_{1310}\Phi_{1310}^{\dagger}\Phi_{221}\Phi_{221}^{\dagger}\} + \lambda_{42} Tr\{\Phi_{1310}\Phi_{1310}^{\dagger}\} Tr\{\Phi_{221}\Phi_{221}^{\dagger}\}$$

$$\mathbf{U}_{2215}^{221} = 32\,\mu_{32} Tr\{\Phi_{2215}\Phi_{2215}^{\dagger}\Phi_{221}\Phi_{221}^{\dagger}\} + \lambda_{32} Tr\{\Phi_{2215}\Phi_{2215}^{\dagger}\} Tr\{\Phi_{221}\Phi_{221}^{\dagger}\}$$

In addition to the above the above crossed terms, the following terms are also possible. They have significant effects on the solution of the minimum. They are responsible for the left-right asymmetry in the vacuum which was previously discussed in § (10.4). We have

$$\begin{aligned}\mathbf{U}_{LR}^{221} &= 32\,\kappa_1\; Tr\{\Phi_{31\bar{1}\bar{0}}^{\dagger}\Phi_{221}\Phi_{1310}^{\dagger}\Phi_{221}\} \\ \mathbf{U}_{LR}^{2215} &= (8/3)\,\kappa_2\; Tr\{\Phi_{1310}^{\dagger}\Phi_{2215}\Phi_{31\bar{1}\bar{0}}^{\dagger}\Phi_{2215}\}\end{aligned} \tag{11.7}$$

By definition the total potential $\mathbf{U}_{Higgs}$ consists of the sum of all the $\mathbf{U}$ terms above in eqs. (11.4) to (11.7). The kinetic terms $\mathbf{T}_{Higgs}$ is defined as

$$\mathbf{T}_{Higgs} = Tr\;[D_\mu\,(\,\Phi_{111}\; + \Phi_{45} + \Phi_{31\bar{1}\bar{0}} + \Phi_{1310} + \;\Phi_{2215} + \;\Phi_{221}\;)]^2 \equiv Tr\;[D_\mu\,(\,\Phi_{all}\,)]^2 \tag{11.8}$$

where D_μ is the covariant gauge derivative. The gauge invariant kinetic term of the Higgs field is

$$Tr\;(D_\mu \Phi_{all})^2 = Tr\;\left(\partial_\mu \Phi_{all}\; + i\,g\,\frac{1}{\sqrt{2}}\,[\Phi_{all}, \Sigma\cdot W]\right)^2 \tag{11.9}$$

Here $(\;\;)^2 = (\;\;)(\;\;)^\dagger$. And g is the single gauge coupling of $SO(10)$. Also $\Sigma \cdot W$ is the gauge term given in eq. (4.1). The Higgs Lagrangian is composed of the the kinetic and the potential terms :

$$\mathbf{L}_{Higgs} = \mathbf{T}_{Higgs} - \mathbf{U}_{Higgs} \tag{11.10}$$

In the following two sections, we will investigate the above potential and kinetic parts separately. The kinetic part will generate the masses for the gauge fields and the potential part will generate the masses of the Higgs fields upon spontaneous symmetry breakdown.

11.2 SSB of the Higgs Lagrangian: The Potential Part

11.2.1 Minimizing the Higgs Potential

We can substitute the vevs given in eq. (11.3) directly into the Higgs potential given in eq. (11.10) and minimize the total Higgs potential $\mathbf{U}_{higgs}$ with respect to $\mathbf{v}_i \equiv \{\mathbf{z}, \mathbf{x}, \mathbf{v_L}, \mathbf{v_R}, \mathbf{u}, \mathbf{v}, \mathbf{k}\}$. To do so one needs the explicit expressions of the various Γ's appearing in eqs. (11.1) and (11.2). These were explicitly studied in sections on the Higgs multiplets and can be borrowed from there. The Higgs potential at the minimum is a very long expression and we will spare our self from presenting it explicitly. We get for $\partial \mathbf{U}_{higgs}/\partial \mathbf{v}_i = 0$, the following set of equations

$$
\begin{aligned}
(i) \quad & 2z^2(\lambda_1+\beta_1) + k^2(\lambda_{12}+\mu_{12}) + (u^2+v^2)(\lambda_{13}+\mu_{13}) \\
& + ({v_L}^2+{v_R}^2)(\lambda_{14}+\mu_{14}) + x^2(\lambda_{15}+\mu_{15}) - \mu_1 = 0 \\[2ex]
(ii) \quad & 2x^2(\lambda_5+\beta_5) + k^2(\lambda_{52}+\mu_{52}) + (u^2+v^2)(\lambda_{53}+\mu_{53}) \\
& + ({v_L}^2+{v_R}^2)(\lambda_{54}+\mu_{54}) + z^2(\lambda_{15}+\mu_{15}) - \mu_5 = 0 \\[2ex]
(iii) \quad & v_R(e^{2i\alpha}k^2\kappa_1 + e^{2i\delta}\,u^2\,\kappa_2) + 2e^{i(\beta+\gamma)}v_L(k^2(\lambda_{42}+\mu_{42}) \\
& + (\,u^2+v^2\,)\,\lambda_{43} + z^2\,(\,\lambda_{14}+\mu_{14}\,) + 2\,{v_L}^2\,(\lambda_4+\beta_4) \\
& - \mu_4 + {v_R}^2\,\mu_{44} + u^2\,\mu_{43} + x^2\,(\,\lambda_{54}+\mu_{54}\,)\,) = 0 \\[2ex]
(iv) \quad & v_L(e^{2i\alpha}k^2\kappa_1 + e^{2i\delta}u^2\kappa_2) + 2e^{i(\beta+\gamma)}v_R(k^2(\lambda_{42}+\mu_{42}) \\
& + (\,u^2+v^2\,)\,\lambda_{43} + {v_L}^2\mu_{44} + 2\,{v_R}^2\,(\,\lambda_4+\beta_4\,) - \mu_4 \\
& + z^2\,(\,\lambda_{14}+\mu_{14}\,) + u^2\,\mu_{43} + x^2\,(\,\lambda_{54}+\mu_{54}\,)\,) = 0 \\[2ex]
(v) \quad & v_L\,v_R\,\kappa_2\,e^{-i(\beta+\gamma-\,2\,\delta\,)} + 2\,(\,u^2+v^2\,)\,\beta_3 - \mu_3 \\
& + 2\,u^2\,\lambda_3 + z^2\,(\lambda_{13}+\mu_{13}\,) + k^2\,(\,\lambda_{32}+\mu_{32}\,) \\
& + ({v_L}^2+{v_R}^2)(\lambda_{43}+\mu_{43}) + x^2(\lambda_{53}+\mu_{53}) = 0 \\[2ex]
(vi) \quad & x^2(\lambda_{53}+\mu_{53}) + (\,{v_L}^2+{v_R}^2)\,\lambda_{43} + z^2(\,\lambda_{13}+\mu_{13}\,) \\
& + 2(u^2+v^2)\beta_3 + k^2(\lambda_{32}+\mu_{32}) + 2\,v^2\,\lambda_3 - \mu_3 = 0 \\[2ex]
(vii) \quad & e^{i(\,2\,\alpha-\beta-\gamma)}\,v_L\,v_R\,\kappa_1 + 2\,k^2\,(\lambda_2+\beta_2) + (\,{v_L}^2+{v_R}^2\,)\,(\,\lambda_{42}+\mu_{42}\,) \\
& + (u^2+v^2)(\lambda_{32}+\mu_{32}) + x^2(\lambda_{52}+\mu_{52}) + z^2(\lambda_{12}+\mu_{12}) - \mu_2 = 0
\end{aligned}
\tag{11.11}
$$

These equations can be solved for the vevs in terms of the Higgs couplings. The easiest way is to divide equation (iii) and equation (iv) by $2e^{i(\beta+\gamma)}v_L$ and $2e^{i(\beta+\gamma)}v_R$ respectively. Then through adding and subtracting these resulting equations they can be solved among themselves for v_L and v_R as below. We have

$$
\begin{aligned}
v_L^2+v_R^2 = & \frac{1/2}{\lambda_4+\beta_4}\,\big[\,\mu_4 - \left(u^2+v^2\right)\,\lambda_{43} - z^2\,\left(\lambda_{14}+\mu_{14}\right) - \\
& \qquad k^2\,\left(\lambda_{42}+\mu_{42}\right) - u^2\,\mu_{43} + x^2\,\left(\lambda_{54}+\mu_{54}\right)\big]\; = 2\,f_1 \\[2ex]
v_L\,v_R = & \frac{e^{i\,(2\,\alpha-\beta-\gamma)}\,k^2\,\kappa_1 + e^{i\,(2\,\delta-\beta-\gamma)}\,u^2\,\kappa_2}{4\,(\,\beta_4+\lambda_4\,) - 2\,\mu_{44}} = 2\,f_2
\end{aligned}
\tag{11.12}
$$

It is seen that the upper equation describes a circle and the lower equation a hyperbola on the plane with the axes v_L and v_R. The solution corresponds to the geometric intersection of the two curves. In terms of the above defined variables f_1 and f_2, we obtain

$$
v_R = \sqrt{2}\sqrt{{f_1}^2 + \sqrt{{f_1}^4 - {f_2}^2}}\quad , \qquad v_L = \sqrt{2}\sqrt{{f_1}^2 - \sqrt{{f_1}^4 - {f_2}^2}} \tag{11.13}
$$

Furthermore, it is appropriate to eliminate v_L and v_R in the remaining 5 equations, i.e., (i), (ii), (v), (vi) and (vii) above by using the expressions in eq. (11.13). Consequently, they become linear in the squares of the vevs. These 5 equations can be further solved for the remaining 5 vevs. Thereby f_1 and f_2 are also determined in terms of the Higgs couplings. However the solutions for $\{\mathbf{z}, \mathbf{x}, \mathbf{u}, \mathbf{v}, \mathbf{k}\}$ will not be presented here since they are very long expressions. An interesting aspect of the above solution is that, if ${f_2}^4 \neq {f_2}^2$, we end up with a left-right asymmetric vacuum amounting to $v_L \neq v_R$.

Yet another aspect of the potential part of the Higgs Lagrangian are the resulting Higgs masses. These can be found through substituting the terms $\mathbf{v}_i + \mathbf{H}_i$ into the Higgs Potential at the minimum. Here $\mathbf{H}_i \equiv \{\mathbf{H_z}, \mathbf{H_x}, \mathbf{H_{v_L}}, \mathbf{H_{v_R}}, \mathbf{H_u}, \mathbf{H_v}, \mathbf{H_k}\}$ are the corresponding Higgs fields associated with the vev $\mathbf{v}_i$. The procedure requires to collect all second order terms like $\mathbf{m}^2_{ij}\, \mathbf{H}_i\, \mathbf{H}_j$ into a mass-squared matrix where $\mathbf{m}^2_{ij}$ are nothing but pre-factors made of the above vevs and the Higgs couplings defined in the Lagrangian. But unfortunately this mass-squared matrix, of size 7×7, has no zero entries and the evaluation of its mass eigenvalues turns out to become very exhaustive. From the other side, one should keep in mind that the coupling strengths in the Higgs Lagrangian are sofar unknown to us. Apart from the theoretical evaluation of the Higgs masses, their existence is currently speculative. Therefore, we do not go into the details of this calculation. More interesting is the kinetic part where the *gauge* boson masses are generated. An apparent advantage of the kinetic part is that the *gauge couplings* enter the mass-squared matrix instead of the *Higgs couplings*.

11.3 SSB of the Higgs Lagrangian: The Kinetic Part

11.3.1 Mass-squared Matrix of the Gauge Fields

In this section, we will mainly consider the term given in eq. (11.9) in its full extent. This term entails the physically most meaningful part of the $SO(10)$ model. We will be able to extract information about how the gauge bosons mix and what masses they receive through the spontaneous symmetry breakdown. The expressions, we arrive at, will depend on the vevs and the coupling strengths of the separate gauge interactions. During the evaluation of the commutator below, the single gauge coupling g of the gauge term should be moved inside it so that g can be replaced by the gauge couplings of the separate gauge interactions which were introduced previously in § (5.27). We have

$$Tr\left(+i\,g\,\frac{1}{\sqrt{2}}\,[\Phi^{vev}_{all}, \Sigma\cdot W]\right)^2 = \mathbf{F}^\dagger M \mathbf{F} \tag{11.14}$$

Here the Higgs fields are collected as a linear sum in Φ_{all} as given in eq. (11.8). Consequently, Φ^{vev}_{all} is the sum of the vevs in eq. (11.3). All the resulting terms arisings from the trace operation can be collected into a mass-squared matrix M where the gauge fields are placed into a column vector F. We have

$$\mathbf{F} = \begin{bmatrix} A_r \\ A_g \\ A_b \\ A'_r \\ A'_g \\ A'_b \\ Y_r \\ Y_g \\ Y_b \\ Y'_r \\ Y'_g \\ Y'_b \\ X_r \\ X_g \\ X_b \\ W^0_L \\ W^+_L \\ W^-_L \\ W^0_R \\ W^+_R \\ W^-_R \\ X_{B-L} \end{bmatrix} \tag{11.15}$$

This column vector has totally 22 entries. But if we count the charge conjugated gauge fields in $F^\dagger$ together, we get altogether 37 gauge fields. Note that $45 - 37 = 8$, correspond to the 8 gluons. These remain massless and are absent in the expression for F. The electromagnetic gauge field is not allowed to get any mass by construction. We expect that $37 - 1 = 36$ gauge fields receive mass. Explicit evaluation of the term in eq. (11.14) yields the mass-squared Matrix M. We have

$$M = \begin{bmatrix}
\alpha_1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \alpha_{11}^* & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \alpha_1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \alpha_{11}^* & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \alpha_1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \alpha_{11}^* & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \alpha_2 & \cdot & \cdot & \alpha_{12}^* & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \alpha_2 & \cdot & \cdot & \alpha_{12}^* & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \alpha_2 & \cdot & \cdot & \alpha_{12}^* & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \alpha_{12} & \cdot & \cdot & \alpha_3 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \alpha_{12} & \cdot & \cdot & \alpha_3 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \alpha_{12} & \cdot & \cdot & \alpha_3 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \alpha_4 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \alpha_4 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \alpha_4 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\alpha_{11} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \alpha_5 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \alpha_{11} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \alpha_5 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \alpha_{11} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \alpha_5 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \alpha_6 & \cdot & \cdot & \alpha_{13}^* & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \alpha_6 & \cdot & \cdot & \alpha_{13} & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \alpha_7 & \cdot & \cdot & \alpha_{14} & \alpha_{15} \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \alpha_{13} & \cdot & \cdot & \alpha_8 & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \alpha_{13}^* & \cdot & \cdot & \alpha_8 & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \alpha_{14} & \cdot & \cdot & \alpha_9 & \alpha_{16} \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \alpha_{15} & \cdot & \cdot & \alpha_{16} & \alpha_{10}
\end{bmatrix}$$

where each dot denotes a zero entry. There are totally 16 independent non-zero entries in M shown with α_i where $i = (1, \ldots, 16)$. These entries are composed as

$$\begin{aligned}
\alpha_1 &= g^2(4u^2 + x^2 + 3(v_L^2 + v_R^2 + z^2))/3 & \alpha_9 &= g_R^2(k^2 + u^2 + v^2 + 4v_R^2)/4 \\
\alpha_2 &= g^2(3k^2 + 3u^2 + 3v^2 + 2x^2 + 6v_R^2 + 6z^2)/6 & \alpha_{10} &= g_{B-L}^2(v_L^2 + v_R^2) \\
\alpha_3 &= g^2(3k^2 + 3u^2 + 3v^2 + 2x^2 + 6v_L^2 + 6z^2)/6 & \alpha_{11} &= 2g^2\sqrt{2/3}\, e^{-i\,(\gamma+\delta)}\, u(e^{+i\,(\gamma+\beta)}v_L + e^{+i\,2\delta}v_R) \\
\alpha_4 &= g^2(4v^2 + x^2 + 3z^2)/3 & \alpha_{12} &= g^2(-3e^{-2i\,\alpha}k^2 - 2\,u\,v\,e^{-i\,(\delta-\theta)})/6 \\
\alpha_5 &= g^2(4u^2 + 4v^2 + 4x^2 + 3v_L^2 + 3v_R^2)/3 & \alpha_{13} &= g_L g_R(k^2\,e^{+2i\,\alpha} - 2\,u\,v\,e^{+i\,(\delta-\theta)})/4 \\
\alpha_6 &= g_L^2(k^2 + u^2 + v^2 + 2v_L^2)/4 & \alpha_{14} &= -g_L g_R(k^2 + u^2 + v^2)/4 \\
\alpha_7 &= g_L^2(k^2 + u^2 + v^2 + 4v_L^2)/4 & \alpha_{15} &= -g_{B-L}g_R\,v_L^2 \\
\alpha_8 &= g_R^2(k^2 + u^2 + v^2 + 2v_R^2)/4 & \alpha_{16} &= -g_{B-L}g_R\,v_R^2
\end{aligned} \tag{11.16}$$

The entries are exact expressions and no approximations were done. The couplings g, g_L, g_R and g_{B-L} are as defined in § (5.27). However their numerical values are subject to renormalization which will be considered later in § 14. Note that α_{11} , α_{12} and α_{13} are complex valued. Therefore the following definitions for phases will be useful

$$\begin{aligned}
\alpha_{13} &= |\alpha_{13}|\ e^{i\,\zeta_1} \\
\alpha_{12} &= |\alpha_{12}|\ e^{i\,\zeta_2} \\
\alpha_{11} &= |\alpha_{11}|\ e^{i\,\zeta_3}
\end{aligned} \tag{11.17}$$

The explicit expressions for ζ_1, ζ_2 and ζ_3 are given in eqs. (11.26) and (11.27). For future reference, we additionally define some new combinations which read

$$\begin{aligned}
\alpha_{20} &= \sqrt{((k^2+u^2+v^2)(v_L^2+v_R^2)+4v_L^2v_R^2)((g_L^2+g_R^2)g_{B-L}^2+g_L^2g_R^2)} \\
\alpha_{21} &= g_L^2(k^2+u^2+v^2-4v_L^2)/4 \\
\alpha_{22} &= g_R^2(k^2+u^2+v^2-4v_R^2)/4
\end{aligned} \tag{11.18}$$

These additional α's will be useful in expressing the mass eigenvalues and mass eigenstates in an equivalent but drastically shorter form. As next, we will work out the mass eigenstates and mass eigenvalues of this mass-squared matrix M.

11.3.2 Gauge Bosons: Mixing and Masses

Mass Eigenstates

The mass matrix M yields 22 eigenstates. But as we pointed out in the former section due to the Hermicity of M, these 22 eigenstates relate us to totally 37 eigenstates. Among them only one eigenstate turns out to have zero mass and should be identified as the gauge field of the residual $U(1)_Q$ symmetry. We have exactly

$$A = e\left(\frac{W_L^0}{g_L}+\frac{W_R^0}{g_R}+\frac{X_{B-L}}{g_{15}}\right) \tag{11.19}$$

Herein the photon appears as a mixture of the above electrically neutral gauge fields. The coupling e normalizes the electromagnetic gauge field A_μ as shown in § (5.27). Three more eigenstates that follow from M are found as

$$\begin{aligned}
W_1^\pm &= e^{\pm i\,\zeta_1}\cos\xi_1\,W_L^\pm+\sin\xi_1\,W_R^\pm \\
Z_1^0 &= \frac{1}{\mathcal{N}}\left(\beta_1\,W_L^0+\beta_2\,W_R^0+\beta_3\,X_{B-L}\right)
\end{aligned} \tag{11.20}$$

where the above mixing parameters ξ_1, β_1, β_2 and β_3 are composed of the gauge couplings and the vevs. The phase ζ_1 is composed *only* of the vevs and the phases that enter the vevs. Indeed the origin of ζ_1 are the phases in the vevs. The mixing parameters $\xi_1, \beta_1, \beta_2, \beta_3$ and the phase ζ_1 will be explicitly presented in eqs. (11.23), (11.24) and (11.26). The electrically charged massive gauge fields $W_1^\pm$ are a mixture of $W_L^\pm$ and $W_R^\pm$. As will be shown later the phase ζ_1 is a source for CP violation. The neutral gauge field Z_1^0 is composed of the same gauge fields that enter A_μ. But is a massive mass eigenstate. $\mathcal{N}$ is introduced to normalize the mass eigenstate Z_1^0. In addition to the Z_1^0 and the $W_1^\pm$ gauge fields, we find 3 more similar mass eigenstates of M. These are

$$\begin{aligned}
W_2^\pm &= -e^{\pm i\,\zeta_1}\sin\xi_1\,W_L^\pm+\cos\xi_1\,W_R^\pm \\
Z_2^0 &= \frac{1}{\mathcal{N}'}\left(\beta_1'\,W_L^0+\beta_2'\,W_R^0+\beta_3'\,X_{B-L}\right)
\end{aligned} \tag{11.21}$$

where the mixing angle ξ_1 appears again, but mixes the $W_L^\pm$ and $W_R^\pm$ fields differently: The $W_1^\pm$ and $W_2^\pm$ mass eigenstates are orthonormal to each other. Z_2^0 mixes through the parameters β_1', β_2' and β_3'. $\mathcal{N}'$ is introduced to normalize the mass eigenstate Z_2^0. These are again composed of the gauge couplings and the vevs. The mixing parameters β_1', β_2' and β_3' will be explicitly presented in eq. (11.25). The remaining massive mass eigenstates are collectively found as

$$\begin{aligned}
W_3 &= \quad Y_\alpha' \quad : \text{no mixing} \\
\\
W_4 &= \quad e^{i\,\zeta_2}\cos\xi_2\,Y_\alpha+\sin\xi_2\,A_\alpha' \\
W_5 &= -e^{i\,\zeta_2}\sin\xi_2\,Y_\alpha+\cos\xi_2\,A_\alpha' \\
\\
W_6 &= \quad e^{i\,\zeta_3}\cos\xi_3\,A_\alpha+\sin\xi_3\,X_\alpha \\
W_7 &= -e^{i\,\zeta_3}\sin\xi_3\,A_\alpha+\cos\xi_3\,X_\alpha
\end{aligned} \tag{11.22}$$

Here the gauge fields Y_α' do not undergo any mixing. This is not an accident. The electric charge of Y' doesn't match with any other of the available gauge fields. We formally denote these Y_α' gauge fields with W_3. It should be kept in mind that W_3 comes in three different colors and is electrically charged. The W_3 mass eigenstates form a color triplet with degenerate mass. We see that Y_α and A'_α mix through the parameter ξ_2 into two different mass eigenstates where ξ_2 is given in eq. (11.23). These are shown with W_4 and W_5 and have different masses. Both W_4 and W_5 are electrically charged color triplets. The masses within the color triplets are $degenerate$. Finally, as was mentioned in § 5.2 the X_α lepto-quark gauge fields mix with A_α's through the parameter ξ_3 into two different mass eigenstates with different masses where ξ_3 is given in eq. (11.23). Here again the W_6 and W_7 gauge fields are electrically charged color triplets and the masses within the color triplets are $degenerate$.

Although we haven't presented the explicit expressions for the masses of the above gauge fields yet, it might be useful to mention qualitatively some remarkable points:

The massive gauge fields $(W_1^\pm, Z_1^0)$ and $(W_2^\pm, Z_2^0)$ should be separated by a sizeable mass gap. It would be phenomenologically consistent to expect such a gap because either the former or the latter "triplet" should correspond to the observed gauge fields $(W^\pm, Z)$ of the electroweak theory. Actually it is the mixing angle ξ_1 which is determinant in this identification. For the above type of mixing among W_L and W_R there will be stringent bounds on ξ_1 [44]. If $\cos\xi_1$ is relatively bigger than $\sin\xi_1$, the field $W_L^\pm$ will be the dominant component in $W_1^\pm$. This dominance could allow us to relate $(W_1^\pm, Z_1^0)$ with $(W^\pm, Z)$. If it turns out to be the opposite case then $(W_2^\pm, Z_2^0)$ should be identified as the $(W^\pm, Z)$. Of course in both cases, it should be investigated to what fraction $W_R^\pm$ enters the mixed mass eigenstates and whether this fraction could be tolerated by the current experimental errors concerning processes involving charged currents [72][73][74][75][76]. From the other side there will be also bounds on the heavier neutral boson Z_2^0 (or Z_1^0) which come from precision experiments on neutral-current processes [40][42][41][43]. Such issues are awaiting us and will be considered in § 15. We find it sufficient to having presented here the mass eigenstates of M and their properties.

Mixing Parameters

The explicit expressions for the the mixing parameters and phases which we introduced in the preceding section will be presented here. We have

$$\xi_1 = \arcsin\left[1 + \left(\frac{\alpha_6 - \alpha_8 + \sqrt{(\alpha_6-\alpha_8)^2 + 4\,|\alpha_{13}|^2}}{2\,|\alpha_{13}|}\right)^2\right]^{-\frac{1}{2}}$$

$$\xi_2 = \arcsin\left[1 + \left(\frac{\alpha_2 - \alpha_3 + \sqrt{(\alpha_2-\alpha_3)^2 + 4\,|\alpha_{12}|^2}}{2\,|\alpha_{12}|}\right)^2\right]^{-\frac{1}{2}} \tag{11.23}$$

$$\xi_3 = \arcsin\left[1 + \left(\frac{\alpha_5 - \alpha_1 + \sqrt{(\alpha_5-\alpha_1)^2 + 4\,|\alpha_{11}|^2}}{2\,|\alpha_{11}|}\right)^2\right]^{-\frac{1}{2}}$$

where $|\quad|$ denotes the absolute value. The mixing parameters entering the electrically neutral mass eigenstate Z_1^0 are

$$\beta_1 = \frac{g_L}{(g_L\alpha_{15} + g_R\alpha_{16})}\left[(\alpha_7 + \alpha_{21} - \alpha_{10}) - \sqrt{(\alpha_7+\alpha_9+\alpha_{10})^2 - \alpha_{20}^2}\right]$$

$$\beta_2 = \frac{g_R}{(g_L\alpha_{15} + g_R\alpha_{16})}\left[(\alpha_9 + \alpha_{22} - \alpha_{10}) - \sqrt{(\alpha_7+\alpha_9+\alpha_{10})^2 - \alpha_{20}^2}\right] \tag{11.24}$$

$$\beta_3 = 1$$

where the appropriate normalization factor $\mathcal{N}$ is defined as $\mathcal{N} = \sqrt{\beta_1^2+\beta_2^2+\beta_2^2}$. The quantities α_{21}, α_{22} and α_{20} appearing above are defined as in eq. (11.18). The mixing parameters entering the electrically neutral mass eigenstate Z_2^0 are

$$
\begin{aligned}
\beta_1' &= \frac{g_L}{(g_L\alpha_{15}+g_R\alpha_{16})}\left[(\alpha_7+\alpha_{21}-\alpha_{10})+\sqrt{(\alpha_7+\alpha_9+\alpha_{10})^2-\alpha_{20}^2}\right] \\
\beta_2' &= \frac{g_R}{(g_L\alpha_{15}+g_R\alpha_{16})}\left[(\alpha_9+\alpha_{22}-\alpha_{10})+\sqrt{(\alpha_7+\alpha_9+\alpha_{10})^2-\alpha_{20}^2}\right] \\
\beta_3' &= 1
\end{aligned}
\tag{11.25}
$$

Here the appropriate normalization factor $\mathcal{N}'$ is defined as $\mathcal{N}'=\sqrt{\beta'^2_1+\beta'^2_2+\beta'^2_3}$. The CP violating phase ζ_1 which appears in $W_1^\pm$ and also in $W_2^\pm$ is composed of the vevs and the phases of the vevs. We have

$$
\zeta_1=-\arctan\left[\frac{k^2\,\sin(2\,\alpha)-2\,u\,v\,\sin(\delta-\theta)}{k^2\cos(2\,\alpha)-2\,u\,v\;\cos(\delta-\theta)}\right] \tag{11.26}
$$

There are two more phases of interest, namely ζ_2 and ζ_3. These appeared in the mass eigenstates W_4, W_5 and W_6, W_7 respectively. The explicit expressions of these phases are

$$
\begin{aligned}
\zeta_2 &= \arctan\left[\frac{3\,k^2\,\sin(2\,\alpha)+2\,u\,v\,\sin(\delta-\theta)}{3\,k^2\cos(2\,\alpha)+2\,u\,v\;\cos(\delta-\theta)}\right] \\
\zeta_3 &= \arctan\left[\frac{v_L\,\sin(\delta-\beta)+v_R\,\sin(\gamma-\delta)}{v_L\,\cos(\delta-\beta)+v_R\,\cos(\gamma-\delta)}\right]
\end{aligned}
\tag{11.27}
$$

Estimated values of the above parameters will be given in § (15).

Mass Eigenvalues

In this section, we present the explicit expressions of the mass eigenvalues of the mass eigenstates of M. Note that no approximations are done. To avoid lengthy expressions, the mass eigenvalues are given in terms of the α's in eqs. (11.16) and (11.18). The photon has zero mass

$$
M_A=0 \tag{11.28}
$$

In the rest of this section, the *squares* of the masses will be given. This simplifies the appearance of the expressions. The squared masses of $W_1^\pm$ and Z_1^0 are respectively

$$
\begin{aligned}
(M_{W_1^\pm})^2 &= \frac{1}{2}\left[(\alpha_6+\alpha_8)-\sqrt{(\alpha_6-\alpha_8)^2+4\,\alpha_{15}^2}\right] \\
(M^2_{Z_1})^2 &= \frac{1}{2}\left[(\alpha_7+\alpha_9+\alpha_{10})-\sqrt{(\alpha_7+\alpha_9+\alpha_{10})^2-\alpha_{20}^2}\right]
\end{aligned}
\tag{11.29}
$$

During the numerical evaluation of the above masses, it is appropriate to factor g_L out and leave in the expression the other gauge couplings as fractions of g_L like g_R/g_L or g_{15}/g_L. These ratios are nothing but the normalization constants given in eq. (5.26). Indeed these are subject to some renormalization procedure which will be studied in § 14. The squared masses of $W_2^\pm$ and Z_2^0 are respectively

$$
\begin{aligned}
(M_{W_2^\pm})^2 &= \frac{1}{2}\left[(\alpha_6+\alpha_8)+\sqrt{(\alpha_6-\alpha_8)^2+4\,\alpha_{15}^2}\right] \\
(M_{Z_2})^2 &= \frac{1}{2}\left[(\alpha_7+\alpha_9+\alpha_{10})+\sqrt{(\alpha_7+\alpha_9+\alpha_{10})^2-\alpha_{20}^2}\right]
\end{aligned}
\tag{11.30}
$$

It is remarkable to see how the $\pm\sqrt{\ldots}$ in the above expressions alternates and produces one heavy and one low massive state each time. Consequently there occurs a *mass gap* between $(W_2^\pm, Z_2^0)$ and $(W_1^\pm, Z_1^0)$. The squared masses of the mass eigenstates in eq. (11.22) are respectively

$$(M_{W_3})^2 = \alpha_4$$

$$(M_{W_4})^2 = \frac{1}{2}\left[(\alpha_2+\alpha_3) - \sqrt{(\alpha_2-\alpha_3)^2 + 4\,|\alpha_{12}|^2}\right] \tag{11.31}$$

$$(M_{W_5})^2 = \frac{1}{2}\left[(\alpha_2+\alpha_3) + \sqrt{(\alpha_2-\alpha_3)^2 + 4\,|\alpha_{12}|^2}\right]$$

$$(M_{W_6})^2 = \frac{1}{2}\left[(\alpha_1+\alpha_5) - \sqrt{(\alpha_1-\alpha_5)^2 + 4\,|\alpha_{11}|^2}\right]$$

$$(M_{W_7})^2 = \frac{1}{2}\left[(\alpha_1+\alpha_5) + \sqrt{(\alpha_1-\alpha_5)^2 + 4\,|\alpha_{11}|^2}\right] \tag{11.32}$$

Similarly due to the $\pm\sqrt{\ldots}$ in the last four expressions above, it is seen that M_{W_4} and M_{W_5} are separated by a mass gap. The same also occurs for M_{W_6} and M_{W_7}.

Numerical estimates of the above masses and as well as the mass gaps and the values of the mixing parameters and phases can only be studied once we know the vevs and the values of the coupling strengths at all mass scales, because the evaluation of the mentioned quantities requires these as an input. But unfortunately we don't have precise knowledge about all the input values:

Note that the Higgs couplings are unknown. Therefore it was not possible to find the values of the vevs from the minimum of the Higgs Potential. Quite similarly the various gauge couplings that are *not contained* in the electroweak theory like g_R and g_{B-L} are also unknown. Nevertheless there are still some techniques that we can use: One possibility is to *fit* the values of the vevs and the values of the gauge couplings, in that we make use of the *experimentally* verified parameters of the electroweak theory like the known masses of the $W^{\pm}$ and the Z bosons and the values of the 3 known coupling strengths at the electroweak mass scale, namely the Fermi scale. Another possibility is to make use of the fact that the various couplings strengths satisfy *gauge coupling unification* at the unification mass scale.

In this respect certain relations among the parameters of the $SO(10)$ theory and the parameters of the electroweak theory will serve as a bridge to estimate the unknown parameters as far as possible. Estimated values of the above masses will be given in § 15.

12. THE YUKAWA SECTOR: QUARK AND LEPTON MASSES

12.1 Quark and Charged Lepton Masses via 126 and 10

In this section, we will construct a suitable Yukawa Lagrangian. For the generation of fermion masses, we should consider again those Higgs fields which entered the Higgs Lagrangian. These where collected in the expression of Φ_{all} in eq. (11.8). But note that the Higgs multiplets defined through Φ_{111} and through Φ_{45} in eq. (11.1) can not produce any fermion masses. Among the Higgs fields collected in Φ_{all}, only $\Phi_{31\bar{1}0}$, Φ_{1310} , Φ_{2215} and Φ_{221} can be used in the Yukawa Lagrangian. To shorten expressions, it is useful to collect the Higgs fields that enter the Yukawa Lagrangian under a single expressions. We define

$$\Phi_Y = \Phi_{31\bar{1}0} + \Phi_{1310} + \Phi_{2215} + \Phi_{221} \tag{12.1}$$

Using this definition for Φ_Y the suitable Yukawa Lagrangian $\mathcal{L}_Y$ can be written down. We have

$$\mathcal{L}_Y = Y_{ij} \ \left(\bar{\Psi}_i \ \Phi_Y \ \Psi_j \right) \ + \ \text{h.c.} \tag{12.2}$$

Here Y_{ij} is by definition the Yukawa coupling. It carries family indices where $i, j = (1, 2, 3)$ and Ψ_j is the family spinor transforming under the $SO(10)$ representation. Note that there is summation of the family space in $\mathcal{L}_Y$. We will assume that all 3 families couple to each other with equal strengths. This leads to the well known form

$$Y_{ij} = \frac{1}{3} \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \tag{12.3}$$

where $1/3$ is an appropriate normalization constant [77]. Note that the entries are just uniformly filled with 1. At this stage, we should also ask whether the isospin up fermions, the isospin down fermions, the charged leptons and the neutral leptons couple to each other all with the *same* strength ? If this is *not* the case then we should introduce new Yukawa couplings like $Y_{ij}^{2/3}$, $Y_{ij}^{1/3}$, Y_{ij}^{+} and Y_{ij}^{0} which are again uniform with respect to family indices but are distinguished with respect to electric charge as indicated in the superscripts. Note that, being up or down in the isospin space manifests itself also through electric charge. But we will disregard all these possibilities for the moment. It would be worth considering it first provided that our assumption of uniformity fails to reproduce the fermion masses of the heaviest generation successfully. Using the definitions of the vevs in eq. (11.3), we arrive at $\langle \Phi_Y \rangle$. Let us substitute $\langle \Phi_Y \rangle$ in $\mathcal{L}_Y$ above. Since the fermions become massive one should consider the mass eigenstates rather than the flavor eigenstates. From the other side only the fermions with equal charge can undergo mixing. Therefore all 3 flavors of say, the up fermions should be collected in a 3 by 3 matrix. Similarly this should also be done for the down fermions, the charged leptons and the neutral leptons where the last one needs special attention since there will also appear Majorana masses for neutral leptons. The mass terms for neutrinos will be separately considered in § (12.2). Since the Yukawa couplings were uniform, so the entries of these 3 by 3 mass matrices for the up fermions, the down fermions and the charged leptons will also be uniform. That means two of the mass eigenvalues will always be zero and only one eigenstate becomes massive. These fermions can be identified as the heaviest fermion generation, namely the third generation. Since the vevs have phases, the quarks will have phases too. We have

$$\begin{aligned} \mathbf{m}_t &= m_t \, e^{i\,\zeta_t} \ , & \mathbf{m}_{\nu_\tau}^D &= m_{\nu_\tau}^D \, e^{i\,\zeta_{\nu_\tau}^D} \\ \mathbf{m}_b &= m_b \, e^{i\,\zeta_b} \ , & \mathbf{m}_\tau &= m_\tau \, e^{i\,\zeta_e} \end{aligned} \tag{12.4}$$

Here D indicates Dirac masses of neutrinos and the various phases of quarks are shown with ζ as above. m_t and m_b are the top and bottom quark masses respectively. $m_{\nu_\tau}^D$ and m_τ are the tau-neutrino and tau masses respectively. We have

$$\begin{aligned} m_t &= \left| + \frac{e^{+i\,\alpha}\,k}{4\sqrt{2}} - \frac{e^{i\,\delta}\,u}{2\sqrt{6}} \right| = \sqrt{\frac{k^2}{32} - \frac{u\,k}{8\sqrt{3}}\,\cos(\alpha - \delta) + \frac{u^2}{24}} \\ m_b &= \left| - \frac{e^{-i\,\alpha}\,k}{4\sqrt{2}} - \frac{e^{i\,\theta}\,v}{2\sqrt{6}} \right| = \sqrt{\frac{k^2}{32} + \frac{v\,k}{8\sqrt{3}}\,\cos(\alpha + \theta) + \frac{v^2}{24}} \end{aligned} \tag{12.5}$$

The mass of the leptons are

$$
\begin{aligned}
m^D_{\nu_\tau} &= \left| + \frac{e^{+i\,\alpha}\,k}{4\sqrt{2}} + \frac{3\,e^{i\,\delta}\,u}{2\sqrt{6}} \right| = \sqrt{\frac{k^2}{32} + \frac{u\,k}{8\sqrt{3}}\,\cos(\alpha-\delta) + \frac{3\,u^2}{8}} \\
m_\tau &= \left| - \frac{e^{-i\,\alpha}\,k}{4\sqrt{2}} + \frac{3\,e^{i\,\theta}\,v}{2\sqrt{6}} \right| = \sqrt{\frac{k^2}{32} - \frac{v\,k}{8\sqrt{3}}\,\cos(\alpha+\theta) + \frac{3\,v^2}{8}}
\end{aligned}
\tag{12.6}
$$

where in the first line the Dirac mass term of the tau-neutrino is given. The final expression of the mass of the tau-neutrino will be obtained in conjunction with the Majorana mass terms that result together from $\mathcal{L}_Y$ in eq. (12.2). As seen in the above expressions, all the fermions in the third generation receive mass via the 10 through the vev k. In contrast, the vev u contributes only to the up fermion and the vevs v contributes only to the down fermion masses. This can naturally induce an asymmetry between the down and up fermion masses. Note also that the phases δ and θ appear in the up- and down-fermions mass terms respectively which can reinforce this asymmetry further. Another remarkable point is that the vev of k and u or similarly k and v are interfering.

Direct evaluation of the above mass terms requires inescapably the values of the vevs and the phases. This will be first achieved in § 15.

12.2 Neutral Lepton Masses via 126 and 10

Let us continue with the Dirac and Majorana mass terms of neutral leptons that result from $\mathcal{L}_Y$ in eq. (12.2). The Dirac mass of the tau-neutrino was independently given as in eq. (12.6). Similarly the Majorana mass terms for the tau-neutrinos read

$$
\begin{aligned}
\mathbf{m}^M_{\nu_{\tau_L}} &= v_L\,e^{i\,\beta} \\
\mathbf{m}^M_{\nu_{\tau_R}} &= v_R\,e^{i\,\gamma}
\end{aligned}
\tag{12.7}
$$

where the superscript M denotes Majorana mass term and L, R denote handedness. Now we will treat these terms together. The Dirac and Majorana mass terms of the neutrinos resulting from $\mathcal{L}_Y$ can be arranged into a 2 by 2 matrix where each entry has a 3 by 3 flavor subspace [78][10], if we collect all these neutrino mass terms under the expression $\mathcal{L}^\nu_Y$. We have

$$
\mathcal{L}^\nu_Y = 2\,\begin{pmatrix} \bar{f}_i & \bar{F}_i \end{pmatrix} \begin{pmatrix} \mathbf{M}^L_{ij} & \mathbf{M}^D_{ij} \\ (\mathbf{M}^D_{ij})^\dagger & \mathbf{M}^R_{ij} \end{pmatrix} \begin{pmatrix} f_j \\ F_j \end{pmatrix} = \begin{pmatrix} \bar{f}_i & \bar{F}_i \end{pmatrix} \mathcal{M} \begin{pmatrix} f_j \\ F_j \end{pmatrix}
\tag{12.8}
$$

Here $\mathcal{M}$ is the neutrino mass matrix. The fermion states f and F are the Majorana-neutrino flavor-eigenbasis. They are defined as

$$
\begin{aligned}
f_j &= \frac{e^{-i\,\beta/2}\nu^j_L + e^{i\,\beta/2}(\nu^j_L)^c}{\sqrt{2}} \qquad & \bar{f}_i &= \frac{e^{+i\,\beta/2}\overline{\nu^i_L} + e^{-i\,\beta/2}\overline{(\nu^i_L)^c}}{\sqrt{2}} \\
F_j &= \frac{e^{-i\,\gamma/2}\nu^j_R + e^{i\,\gamma/2}(\nu^j_R)^c}{\sqrt{2}} \qquad & \bar{F}_i &= \frac{e^{+i\,\gamma/2}\overline{\nu^i_R} + e^{-i\,\gamma/2}\overline{(\nu^i_R)^c}}{\sqrt{2}}
\end{aligned}
\tag{12.9}
$$

Here $i, j = (1, 2, 3)$. The phases in f and F are identified as β and γ respectively. These originate from the vevs in eq. (11.3). Note also that $f^c = f$ and $F^c = F$. By means of the above eigenbasis, the 3 by 3 mass matrices in $\mathcal{M}$ read

$$
\begin{aligned}
\mathbf{M}^D_{ij} &= Y_{ij}\,\left(\frac{e^{+i\,\alpha}\,k}{4\sqrt{2}} + \frac{3\,e^{i\,\delta}\,u}{2\sqrt{6}}\right) \\
\mathbf{M}^L_{ij} &= Y_{ij}\,v_L \\
\mathbf{M}^R_{ij} &= Y_{ij}\,v_R
\end{aligned}
\tag{12.10}
$$

It can be easily checked that the Majorana mass terms in eq. (12.7) are generated through the product $f_i\bar{f}_j\,\mathbf{M}^L_{ij}$ and $F_i\bar{F}_j\,\mathbf{M}^R_{ij}$ respectively. The Dirac mass terms of the neutrinos are generated through $\bar{F}_i\,f_j\,\mathbf{M}^D_{ij}$ or $\bar{f}_i\,F_j\,\mathbf{M}^D_{ij}$. These two terms should correctly yield the Dirac mass term in eq. (12.6). Consequently the phases β and γ should satisfy some relation for consistency. We have $e^{i(\beta-\gamma)/2} = 1$. The CP violating phase ζ_1 in eq. (11.26) does not become trivial through this condition. Assume $\beta - \gamma = 2\,\pi\,n$ where n is an integer number, then $\tan\zeta_3 = -\cot\beta$

provided that $v_L \neq v_R$. If we diagonalize $\mathcal{M}$, 4 mass eigenstates will have zero masses because of the uniform entries in Y_{ij}. The remaining two mass eigenstates, which should be identified as the mass eigenstates of the tau-neutrino of the third generation, have non-zero masses. These masses are

$$\begin{aligned} m_{\nu_{\tau_1}} &= v_L + v_R - \frac{1}{2\sqrt{2}}\sqrt{k^2 + 12\,u^2 + 8\,(v_L - v_R)^2 + 4\sqrt{3}\,k\,u\,\cos(\alpha - \delta)} \\ m_{\nu_{\tau_2}} &= v_L + v_R + \frac{1}{2\sqrt{2}}\sqrt{k^2 + 12\,u^2 + 8\,(v_L - v_R)^2 + 4\sqrt{3}\,k\,u\,\cos(\alpha - \delta)} \end{aligned} \tag{12.11}$$

Here $m_{\nu_{\tau_1}}$ and $m_{\nu_{\tau_2}}$ are the masses of the mass eigenstates ν_{τ_1} and ν_{τ_2} respectively. These are not pure in the Majorana flavor eigenbasis f_i or F_i, i.e., they have no pure handedness but are mixtures of f_i and F_i. The mixing can be stated over a mixing angle ξ_4 and a phase ζ_4 where the latter is again a possible source for CP violation [79]. The mass eigenstates for $m_{\nu_{\tau_1}}$ and $m_{\nu_{\tau_2}}$ read

$$\begin{aligned} \nu_{\tau_1} &= +e^{i\,\zeta_4} f_3 \cos\xi_4 + F_3 \sin\xi_4 \\ \nu_{\tau_2} &= -e^{i\,\zeta_4}\, f_3 \sin\xi_4 + F_3 \cos\xi_4 \end{aligned} \tag{12.12}$$

The phase ζ_4 which follows from $\mathcal{M}$ is found as

$$\zeta_4 = \arctan\left[-\frac{k\,\sin(\alpha) + u\,\sqrt{12}\,\sin(\delta)}{k\,\cos(\alpha) + u\,\sqrt{12}\,\cos(\delta)}\right] \tag{12.13}$$

Here we have used the letter ζ_4. Because ζ_i with $i = (1, 2, 3)$ are the phases that enter the expressions for mixing of gauge bosons which were already given in eqs. (11.26) and (11.27). Here the mixing parameter ξ_4 is a function of the vevs and their phases. We have

$$\xi_4 = \arcsin\left[\frac{\left(4(v_L - v_R) - \sqrt{2}\sqrt{k^2 + 12\,u^2 + 4\sqrt{3}\,k\,u\,\cos(\alpha - \delta) + 8\,(v_L - v_R)^2}\right)^2}{2\,k^2 + 12\,u^2} + 1\right]^{-\frac{1}{2}} \tag{12.14}$$

In the state of the art, we have left-handed neutrinos in nature which are treated in the framework of the electroweak theory. Recent findings tell us that these left-handed neutrinos have tiny masses [80][81][82][83]. On the other side the right-handed neutrinos have never been observed. The electroweak theory naturally excludes the existence of right-handed neutrinos. However we can not write down any Dirac mass term for the neutrinos in the framework of the electroweak theory which would lead right-handed neutrinos to have the same mass with left-handed neutrinos. This would be in conflict with the absence of light right-handed neutrinos. Obviously the electroweak theory has a shortcoming in the neutrino sector.

In the $SO(10)$ theory the above mass matrix $\mathcal{M}$ can generate unequal masses for left-handed and right-handed neutrinos. Consequently almost massless left-handed neutrinos and very massive right-handed neutrinos can co-exist in nature without conflicting the current status of experiments.

Now in the light of our analysis, it would be consistent to identify ν_{τ_1} in eq. (12.12) with the heaviest observed left-handed neutrino which is the tau-neutrino. Consequently ν_{τ_2} in eq. (12.12) should then correspond to the hypothetical right-handed tau-neutrino. But nor ν_{τ_1} neither ν_{τ_2} has pure handedness which is revealed by the above mixing in eq. (12.12). Therefore we should expect that $\sin\xi_4$ equals almost zero in the above mixing so that ν_{τ_1} becomes almost a pure left-handed mass eigenstate and ν_{τ_2} becomes almost a pure right-handed mass eigenstate. As a result, the above described mixing would be physically acceptable only if $\sin\xi_4$ and $m_{\nu_{\tau_1}}$ are very small. The evaluation of $\sin\xi_4$ and $m_{\nu_{\tau_1}}$ will be postponed until we gain some accurate knowledge of the vevs in § 15 via a *fitting* procedure.

13. COUPLING OF THE GAUGE FIELDS TO FERMIONS

13.1 Charged Currents (CC)

The Φ_{221} and Φ_{2215} Higgs fields given in eq. (11.3) gave rise to a complex mixing among the $W_L^{\pm}$ and $W_R^{\pm}$ gauge fields. The resulting mixing was previously stated in eqs. (11.20) and (11.21). It was found that $W_1^{\pm}$ and $W_2^{\pm}$ are corresponding to mass eigenstates with different masses as shown in eqs. (11.29) and (11.30). This mixing can be described through a rotation. We have

$$\begin{bmatrix} W_1^{\pm} \\ \\ W_2^{\pm} \end{bmatrix} = \begin{bmatrix} e^{\pm\, i\, \zeta_1} \cos\xi_1 & \sin\xi_1 \\ \\ -e^{\pm\, i\, \zeta_1} \sin\xi_1 & \cos\xi_1 \end{bmatrix} \begin{bmatrix} W_L^{\pm} \\ \\ W_R^{\pm} \end{bmatrix} \tag{13.1}$$

where ξ_1 is the mixing angle and ζ_1 is the phase originating from the vevs which can be traced back to the mass-squared matrix of the gauge fields in § 11.3. The phase is a source for spontaneous CP violation. On the other side $W_1^{\pm}$ and $W_2^{\pm}$ correspond to the physical fields. Therefore we should rewrite the Lagrangian in eq. (5.18) which describes the interaction of fermions with the charged currents, in terms of the physical fields $W_1^{\pm}$ and $W_2^{\pm}$. We expect the original Lagrangian to assume the form given on the right hand side as below. This can be achieved through the inverse transformation in eq. (13.1). We have

$$\mathcal{L}^{CC} = +i\,\sqrt{2}\left(g_R\, J_R^{\pm} \cdot W_R^{\pm} + g_L\, J_L^{\pm} \cdot W_L^{\pm}\right) = +i\,\sqrt{2}\left(g_1\, J_1^{\pm} \cdot W_1^{\pm} + g_2\, J_2^{\pm} \cdot W_2^{\pm}\right) \tag{13.2}$$

Herein g_L and g_R are the coupling strengths of $SU(2)_L$ and $SU(2)_R$ respectively. $J_R^{\pm}$ and $J_L^{\pm}$ are the left-isospin and right-isospin charged currents respectively. By using the inverse transformation above, we obtain

$$\begin{aligned} \mathcal{L}^{CC} = +i\, \frac{1}{\sqrt{2}}\, \{ \bar{f}_u \gamma_\mu \big[& \left(g_L\, \cos\xi_1\, V_L\, P_L - g_R\, e^{i\zeta_1}\, \sin\xi_1\, V_R\, P_R\right)\, W_1^{\mu} \\ & + \left(g_L\, \sin\xi_1\, V_L\, P_L + g_R\, e^{i\zeta_1}\, \cos\xi_1\, V_R\, P_R\right)\, W_2^{\mu} \big]\, f_d + h.c. \} \end{aligned} \tag{13.3}$$

Herein the charged isospin currents are projected through the projection operators P_L and P_R into $J_L^{\pm}$ and $J_R^{\pm}$ respectively where $P_{L,R} = (1 \mp \gamma_5)/2$. Also f_u and f_d denote the up and down fermions. That means we have indeed two Lagrangians written in expression: one for the leptonic sector and one for the quark sector. Indeed the family spinor Ψ in eq. (5.18) contains both quarks and leptons. Consequently we should do the replacements $f_u \to u$ and $f_d \to d$ for quarks, and we should do the replacements $f_u \to \nu$ and $f_d \to e$ for leptons in $\mathcal{L}^{CC}$. On the other side the fermions should also correspond to the physical states and not the flavor states because they become massive through the spontaneous symmetry breakdown initiated through the same Higgs scalar. The mixing among the fermions for each sector is achieved through the V_L and V_R matrices which should be identified as the CKM matrices in case of quarks [84][6][85]. The V_L and V_R matrices should be identified as the Maki-Nagawaka-Sakata (MNS) matrices in case of leptons [86][87][88][89]. Note that the mixing among left- and right-handed fermions might be different therefore we have V_L and also V_R.

13.2 Neutral Currents (NC)

The Φ_{221}, Φ_{2215}, Φ_{1310} and $\Phi_{3,1,\overline{10}}$ Higgs fields given in eq. (11.3) gave rise to a mixing among the W_L^0, W_R^0 and X_{B-L} gauge fields. The resulting mixing was previously stated in eqs. (11.19), (11.21) and (11.20). It was found that Z_1 and Z_2 are corresponding to mass eigenstates with different masses as shown in eqs. (11.29) and (11.30). And A_μ was the massless electromagnetic gauge field. This mixing can be described through the following transformation. We have

$$\begin{bmatrix} A \\ \\ Z_1 \\ \\ Z_2 \end{bmatrix} = \begin{bmatrix} \frac{e}{g_L} & \frac{e}{g_R} & \frac{e}{g_{15}} \\ \\ \frac{\beta_1}{\mathcal{N}} & \frac{\beta_2}{\mathcal{N}} & \frac{\beta_3}{\mathcal{N}} \\ \\ \frac{\beta_1'}{\mathcal{N}'} & \frac{\beta_2'}{\mathcal{N}'} & \frac{\beta_3'}{\mathcal{N}'} \end{bmatrix} \begin{bmatrix} W_L^0 \\ \\ W_R^0 \\ \\ X_{B-L} \end{bmatrix} \tag{13.4}$$

where $\beta_1, \beta_2, \beta_3$ and $\beta'_1, \beta'_2, \beta'_3$ are the mixing parameters which are explicitly given in eqs. (11.24) and (11.25). Furthermore g_L, g_R and g_{B-L} are the coupling strengths of the $SU(2)_L$, $SU(2)_R$ and $U(1)_{B-L}$ gauge interactions respectively and $e, \mathcal{N}$ and $\mathcal{N}'$ are normalization constants where the former is the coupling strength of $U(1)_Q$. The interaction of the fermions with the neutral currents J_{B-L}, J_L^0 and J_R^0 in $SO(10)$ were given in the Lagrangians in eqs. (5.8) and (5.18). These terms can be collected in a Lagrangian as below. We have

$$\begin{aligned} \mathcal{L} &= +i \left(g_L\, J_L^0\, W_L^0 + g_R\, J_R^0\, W_R^0 + \sqrt{\frac{2}{3}}\, g_{B-L}\, J_{B-L} X_{B-L} \right) \\ &= +i\ (e\, J_Q\, A + g_1\, J_{Z_1}\, Z_1 + g_2\, J_{Z_2}\, Z_2) \end{aligned} \tag{13.5}$$

It is appropriate to express this Lagrangian in terms of the physical fields A, Z_1 and Z_2 as described in the second line above. Using the inverse transformation in eq. (13.4), we should obtain

$$\begin{aligned} &\mathcal{L} = \mathcal{L}^{em} + \mathcal{L}^{NC} = \\ &\quad + i \left\{ e\ \left(\bar{f} \gamma_\mu\, Q_f\, f\right)\ A^\mu + \left[\bar{f} \gamma_\mu\, \frac{1}{2}\, \left(c_V^f - c_A^f \gamma_5 \right)\, f \right]\ Z_1^\mu + \left[\bar{f} \gamma_\mu\, \frac{1}{2}\, \left(c_{V'}^f - c_{A'}^f \gamma_5 \right)\, f \right]\ Z_2^\mu + h.c. \right\} \end{aligned} \tag{13.6}$$

where f are the fermions. i.e., quarks and leptons. Q_f is the electric charge of the fermion f. The first term in $\mathcal{L}$ is identified as the electromagnetic interaction Lagrangian $\mathcal{L}^{em}$. The second and the third terms in $\mathcal{L}$ are identified as the neutral current interaction Lagrangian $\mathcal{L}^{NC}$ and are written in a compact form: We introduced two pairs of vector and axial-vector couplings (c_A^f, c_V^f) and $(c_{A'}^f, c_{V'}^f)$ which are composed of the elements of the transformation matrix in eq. (13.4). The vector and axial-vector couplings depend on the vevs and the coupling strengths which can be traced back to the mass-squared matrix of the gauge fields in § 11.3. The vector and axial-vector couplings will also depend on the $(B-L)$, L_3 and R_3 quantum numbers of the fermions f. We will give the expressions for the vector and axial-vector couplings of $SO(10)$ and their values in § 15.3. In the above Lagrangian, it is appropriate to identify the second term with the usual NC interaction of the electroweak theory and the third term as a new NC interaction. If Z_2 has a comparable mass to Z_1 there will be stringent bounds on the $c_{A'}^f, c_{V'}^f$ couplings and the Z_2 mass [41][42][43].

13.3 Charged and 4-Colored Currents

The Φ_{221}, Φ_{2215}, Φ_{1310} and $\Phi_{3,1,\overline{10}}$ Higgs fields given in eq. (11.3) gave rise to a mixing among the Y_α and A'_α fields and a further mixing among the A_α and X_α fields. The Y'_α fields didn't mix with any other gauge field. The resulting mass eigenstates were previously given in eqs. (11.22). The masses of these mass eigenstates were given in eq. (11.31). The mass eigenstates W_4, W_5, W_6, and W_7 can be described through the following rotations. We have

$$W_3 = \quad Y'_\alpha \quad : \text{no mixing} \tag{13.7}$$

$$\begin{bmatrix} W_4 \\ W_5 \end{bmatrix} = \begin{bmatrix} e^{+i\,\zeta_2} \cos\xi_2 & \sin\xi_2 \\ -e^{+i\,\zeta_2} \sin\xi_2 & \cos\xi_2 \end{bmatrix} \begin{bmatrix} Y_\alpha \\ A'_\alpha \end{bmatrix} \tag{13.8}$$

$$\begin{bmatrix} W_6 \\ W_7 \end{bmatrix} = \begin{bmatrix} e^{+i\,\zeta_3} \cos\xi_3 & \sin\xi_3 \\ -e^{+i\,\zeta_3} \sin\xi_3 & \cos\xi_3 \end{bmatrix} \begin{bmatrix} A_\alpha \\ X_\alpha \end{bmatrix} \tag{13.9}$$

From the other side the Lagrangian describing the interactions of fermions with the currents mediated by the $A_\alpha, A'_\alpha, Y_\alpha, Y'_\alpha$ and X_α fields prior to any SSB were given in eqs. (5.1) and (5.8) respectively. We collect these interaction terms in a single Lagrangian. We have

$$\begin{aligned} \mathcal{L} &= +i\, g\sqrt{2}\ \left(J^A \cdot A + J^{A'} \cdot A' + J^Y \cdot Y + J^{Y'} \cdot Y' + J^{X_\alpha} \cdot X_\alpha + h.c. \right) \\ &= +i\, g\sqrt{2}\ (J_4 \cdot W_4 + J_4 \cdot W_4^\mu + J_5 \cdot W_5^\mu + J_6 \cdot W_6^\mu + J_7 \cdot W_7 + h.c.) \end{aligned} \tag{13.10}$$

where we have restated $\mathcal{L}$ in terms of the physical gauge fields W_3, W_4, W_5, W_6, and W_7 on the right hand side above. Using the transformations given in eqs. (13.7), (13.8) and (13.9), we arrive at

$$\begin{aligned}\mathcal{L} = +i\,g\sqrt{2}\ &\Big\{\left(J^A_\mu\, e^{-i\,\zeta_2}\,\cos\xi_2 - J^X_\mu\,\sin\xi_2\right)\,W^\mu_4 + \left(J^A_\mu\, e^{-i\,\zeta_2}\,\sin\xi_2 + J^X_\mu\,\cos\xi_2\right)\,W^\mu_5 \\ &+\left(J^Y_\mu\, e^{-i\,\zeta_3}\,\cos\xi_3 - J^{A'}_\mu\,\sin\xi_3\right)\,W^\mu_6 + \left(J^Y_\mu\, e^{-i\,\zeta_3}\,\sin\xi_3 + J^{A'}_\mu\,\cos\xi_3\right)\,W^\mu_7 \\ &+J^{Y'}_\mu\,W^\mu_3 + h.c.\Big\}\end{aligned} \tag{13.11}$$

Single interaction terms in the above Lagrangian can be found by using the explicit expressions for the currents in eq. (5.4) to (5.7) and eq. (5.11). We have

$$\begin{aligned}
\mathcal{L}_3 &= +\,i\,g\frac{1}{\sqrt{2}}\ \{[\epsilon_{\alpha\beta\gamma}\,(q_u)_\alpha\,\gamma_\mu\,(q^c_u)_\beta + \ (q_d)_\gamma\,\gamma_\mu\,l_d + \ l^c_d\,\gamma_\mu\,(q_d)_\gamma]\,(W^\mu_3)_\gamma + h.c.\} \\
\mathcal{L}_4 &= +\,i\,g\frac{1}{\sqrt{2}}\ \{\left[-(\bar{q}^c_d)_\gamma\,\gamma_\mu\,l^c_d - (\bar{q}^c_u)_\gamma\,\gamma_\mu\,l^c_u + \bar{l}_d\,\gamma_\mu\,(q_d)_\gamma + \bar{l}_u\,\gamma_\mu\,(q_u)_\gamma + \bar{l}^c_u\,\gamma_\mu\,(q_u)_\gamma + \epsilon_{\alpha\beta\gamma}\,(\bar{q}_d)_\alpha\,\gamma_\mu\,(q^c_d)_\beta \right. \\
&\qquad \left. -(\bar{q}^c_u)_\gamma\,\gamma_\mu\,l_u\right]\left[e^{-i\,\zeta_2}\,\cos\xi_2 - \sin\xi_2\right](W^\mu_4)_\gamma + h.c.\} \\
\mathcal{L}_5 &= +\,i\,g\frac{1}{\sqrt{2}}\ \{\left[-(\bar{q}^c_d)_\gamma\,\gamma_\mu\,l^c_d - (\bar{q}^c_u)_\gamma\,\gamma_\mu\,l^c_u + \bar{l}_d\,\gamma_\mu\,(q_d)_\gamma + \bar{l}_u\,\gamma_\mu\,(q_u)_\gamma + \bar{l}^c_u\,\gamma_\mu\,(q_u)_\gamma + \epsilon_{\alpha\beta\gamma}\,(\bar{q}_d)_\alpha\,\gamma_\mu\,(q^c_d)_\beta \right. \\
&\qquad \left. -(\bar{q}^c_u)_\gamma\,\gamma_\mu\,l_u\right]\left[e^{-i\,\zeta_2}\,\sin\xi_2 + \cos\xi_2\right](W^\mu_5)_\gamma + h.c.\} \\
\mathcal{L}_6 &= +\,i\,g\frac{1}{\sqrt{2}}\ \{\left[-\epsilon_{\alpha\beta\gamma}\,(\bar{q}_u)_\alpha\,\gamma_\mu\,(q^c_d)_\beta + \bar{l}^c_u\,\gamma_\mu\,(q_d)_\gamma - (\bar{q}^c_u)_\gamma\,\gamma_\mu\,l_d - \epsilon_{\alpha\beta\gamma}\,(\bar{q}_d)_\alpha\,\gamma_\mu\,(q^c_u)_\beta + \bar{l}^c_d\,\gamma_\mu\,(q_u)_\gamma \right. \\
&\qquad \left. -(\bar{q}^c_d)_\gamma\,\gamma_\mu\,l_u\right]\left[e^{-i\,\zeta_3}\,\cos\xi_3 - \sin\xi_3\right]\,(W^\mu_6)_\gamma + h.c.\} \\
\mathcal{L}_7 &= +\,i\,g\frac{1}{\sqrt{2}}\ \{\left[-\epsilon_{\alpha\beta\gamma}\,(\bar{q}_u)_\alpha\,\gamma_\mu\,(q^c_d)_\beta + \bar{l}^c_u\,\gamma_\mu\,(q_d)_\gamma - (\bar{q}^c_u)_\gamma\,\gamma_\mu\,l_d - \epsilon_{\alpha\beta\gamma}\,(\bar{q}_d)_\alpha\,\gamma_\mu\,(q^c_u)_\beta + \bar{l}^c_d\,\gamma_\mu\,(q_u)_\gamma \right. \\
&\qquad \left. -(\bar{q}^c_d)_\gamma\,\gamma_\mu\,l_u\right]\left[e^{-i\,\zeta_3}\,\sin\xi_3 + \cos\xi_3\right]\,(W^\mu_7)_\gamma + h.c.\}
\end{aligned} \tag{13.12}$$

where $\mathcal{L} = \mathcal{L}_3 + \cdots + \mathcal{L}_7$. Also q and l denote quarks (u, c, t, d, s, b) and leptons $(e, \mu, \tau, \nu_e, \nu_\mu, \nu_\tau)$ respectively. The subscripts u and d denote up and down states respectively. $\epsilon_{\alpha\beta\gamma} = -\epsilon_{\alpha\gamma\beta} = 1$ and the indices (α, β, γ) denote $SU(3)_c$.

Before we end this section, we find it appropriate to mention a few aspects of the $SO(10)$ theory which are linked to cosmology. Actually it is intriguing how grand unified theories of elementary particles and their interactions are connected with cosmology. As it is well known, our universe is predominantly made of matter and we have evidence for that there are more particles than anti-particles [90]. A suggestion from Yoshimura is that the baryon number violation can combine with CP violation to produce a calculable net amount of baryon number even though the universe was initially baryon neutral [91][92].

The mechanism becomes more transparent if one compares the partial decay rates of heavy gauge bosons and anti-gauge bosons (or Higgs bosons) into quark + quark pairs and anti-quark + anti-quark pairs respectively [93]. This can be stated as

$$\frac{\Gamma\left\{(2,2,3) \rightarrow q + q\right\}}{\Gamma\left\{(2,2,3) \rightarrow \text{all}\right\}} \neq \frac{\Gamma\left[(2,2,\bar{3}) \rightarrow \bar{q} + \bar{q}\right\}}{\Gamma\left\{(2,2,\bar{3}) \rightarrow \text{all}\right\}} \tag{13.13}$$

where the $(2, 2, 3)$ multiplet denotes the $A_\alpha, A'_\alpha, Y_\alpha$ and Y' gauge bosons. Similarly the $(2, 2, 3)$ and the $(2, 2, \bar{3})$ gauge bosons can also decay into quark + lepton or anti-quark + anti-lepton pairs respectively [94]. This can be stated as

$$\frac{\Gamma\left\{(2,2,3) \rightarrow q + l\right\}}{\Gamma\left\{(2,2,3) \rightarrow \text{all}\right\}} \neq \frac{\Gamma\left[(2,2,\bar{3}) \rightarrow \bar{q} + \bar{l}\right\}}{\Gamma\left\{(2,2,\bar{3}) \rightarrow \text{all}\right\}} \tag{13.14}$$

To reach from an initially matter anti-matter symmetric universe a universe with net excess of baryons surrounded by a huge number of released photons, we would need B, C and CP violating interactions where B denotes baryon-number. All the vertices above in $\mathcal{L}_3, \ldots, \mathcal{L}_7$ violate separately B and L number but conserve $B - L$ except the vertices in $\mathcal{L}_4$ and $\mathcal{L}_5$ which also violate $B - L$ number. The general requirements for a net excess of baryons in our universe was studied by Sakharov in 1967 [95]. The well know ratio reads

$$\frac{n_B}{n_\gamma} = \frac{n_q - n_{\bar{q}}}{n_q + n_{\bar{q}}} \tag{13.15}$$

where n_q and $n_{\bar{q}}$ are the number of quarks and anti-quarks as products of the above decays. n_B and n_γ are the net baryon and photon numbers. Herein n_q and $n_{\bar{q}}$ can only be different if the interactions are C violating which breaks the symmetry among $n_q = n_{\bar{q}}$. From the other side the same interactions must also be CP violating because n_q and $n_{\bar{q}}$ remain unchanged over a parity transformation [94]. n_γ in the denominator is related with the decays of $q\,\bar{q}$ mesons into photons and the nominator n_B is related with excess quarks that confine to make the baryonic matter around us.

We have previously shown in § 4.3.2 that $SO(10)$ interactions are separately C, P and CP invariant. The spontaneous breakdown of the $SO(10)$ symmetry gave rise to the ζ_2 and ζ_3 phases which are possible sources for CP violation in the above interaction Lagrangians. These phases are explicitly given eq. (11.27). They will be evaluated later in § 15. On the other side the interactions in $\mathcal{L}^{CC}$ which are given in eq. (13.2) are also CP violating due to the ζ_1 phase which is given in eq. (11.26). The value of ζ_1 will be estimated later in § 15.

As we will show later in § 14 and § 15 the vevs v_L and v_R are not equal. This has certain consequences: The $W_1^\pm$ and $W_2^\pm$ gauge bosons acquire different masses and interact also with different strengths. As a result C and P invariance in $\mathcal{L}^{CC}$ is numerically lost.

As a result $SO(10)$ grand unification has the necessary ingredient to produce a net excess of matter over anti-matter. From the $3K$ radiation background and the average density of matter in the universe which roughly equals 10^{-31} g cm^{-3}, the above ratio turns out to read

$$\frac{n_B}{n_\gamma} \approx 10^{-9} \tag{13.16}$$

This value puts a stringent bound on our model [94]. A quantitative analysis of the above ratio using the decay rates is however very complicated and will not be investigated any further.

14. RENORMALIZATION OF THE COUPLING STRENGTHS IN $SO(10)$

We have shown in § 3 that all of the Standard Model gauge interactions described by the direct product gauge group $SU(3)_C \times SU(2)_L \times U(1)_Y$ are derivable from $SO(10)$ gauge interactions. Furthermore, $SO(10)$ grand unification disposes of a single coupling strength g, which formally describes a single force, which is not achieved in the Standard Model. But we should keep in mind that the gauge couplings strengths assigned to the color, weak isospin and hypercharge gauge groups in the Standard Model are not numerically related to each other exactly through the ratios that we stated in eq. (5.26). That means the measured values of these couplings are not satisfying the simple ratios originating from the $SO(10)$ group. This is a well known problem studied in the framework of renormalization [30][96] [97][98]. In the remaining part, we will study the renormalization of these various coupling strengths defined in eq. (5.26).

Let us start with the lucid identification that the separate coupling strengths g_i assigned to the various subgroups of $SO(10)$ are equal in value at some grand unification mass scale M_G whose value is subject to further determination. We have

$$g = \underbrace{\frac{g_L}{C_L}}_{g_1} = \underbrace{\frac{g_R}{C_R}}_{g_2} = \underbrace{\frac{g_{B-L}}{C_{B-L}}}_{g_3} = \underbrace{\frac{g_Y}{C_Y}}_{g_4} = \underbrace{\frac{\sqrt{4\pi\alpha_s}}{C_s}}_{g_5} = \underbrace{\frac{\sqrt{4\pi\alpha_{s'}}}{C_{s'}}}_{g_6} = \underbrace{\frac{e}{C_Q}}_{g_7} \tag{14.1}$$

The renormalization procedure influences the coupling strengths g_i to evolve differently as we move towards lower energy scales. Formally, these unequal coupling strengths can be perceived as the origin of the separate interactions manifesting themselves in nature. One of the fundamental aspects of grand unification is to establish the above equalities among these gauge couplings with the gauge coupling g. For small values of the unification gauge coupling the renormalization equations can be stated as

$$\frac{1}{g_i^2(\mu)} = \frac{1}{g_i^2(Q)} + 2b_i \ln\frac{Q}{\mu} + \sum \frac{b_{ij}^1}{b_j} \ln\frac{Q}{\mu} + \cdots \tag{14.2}$$

where $g_i(\mu)$ and $g_i(Q)$ is the measured strength of g at the energy scale μ and Q respectively [23]. Formally the renormalization of any of the gauge coupling g_i depends on the dimension of the unitary gauge group to which the coupling is assigned. This property is contained in the b_i functions which get contributions from gauge bosons, fermion loops and scalar bosons [31]. If only the gauge bosons and fermion loop contributions to vacuum polarization are taken into account one obtains

$$b_N = \frac{1}{(4\pi)^2}\left[-\frac{11}{3}N + \frac{4}{3}n_g\right] \tag{14.3}$$

where N is related to $SU(N)$ and n_g indicating the number of fermion generations [93][25]. Let us introduce for each of the following subgroup a coupling strength g_i and a function b_N with an appropriate label. We have

$$\begin{array}{llrcl}
U(1)_Q\ , & b_1^{em}\ , & e(Q) & = & C_Q\, g_7(Q) = \sqrt{4\pi\alpha(Q)} \\
SU(4)_C\ , & b_4\ , & \alpha_{s'}(Q) & = & \frac{g_6^2(Q)}{4\pi} \\
SU(3)_C\ , & b_3\ , & \alpha_s(Q) & = & \frac{g_5^2(Q)}{4\pi} \\
U(1)_Y\ , & b_1\ , & g_Y(Q) & = & C_Y\, g_4(Q) \\
U(1)_{B-L}\ , & b_1^c\ , & g_{B-L}(Q) & = & C_{B-L}\, g_3(Q) \\
SU(2)_R\ , & b_2^R\ , & g_R(Q) & = & C_R\, g_2(Q) \\
SU(2)_L\ , & b_2^L\ , & g_L(Q) & = & C_L\, g_1(Q)
\end{array} \tag{14.4}$$

Note that these conventions confirm those in eq. (14.1). Before we start using the above definitions in the renormalization procedure, let us consider certain ratios among the coupling strengths that will be later useful. A well known ratio follows from the electroweak theory. We have

$$\frac{g_Y(Q)}{g_L(Q)} = \frac{C_Y}{C_L}\frac{g_4(Q)}{g_1(Q)} = \tan\theta_L \tag{14.5}$$

where Q indicates the energy scale dependence at which the interaction is probed and $\sin\theta_L(Q)$ is the weak mixing angle. In contrast to the electroweak theory the electromagnetic gauge field is a mixture of the W_L, W_R and X_{B-L} fields in $SO(10)$ as shown in eq. (11.19). Therefore $\sin\theta_L$ does not correspond to any mixing angle in $SO(10)$. But it is still important to us, because the ratio is experimentally determined and can be used as an input in the fitting procedure that will be introduced in § 15. The ratio between the coupling strength g_R of $SU(2)_R$ and the g_{B-L} of $U(1)_{B-L}$ provides a similar use. We have

$$\frac{g_{B-L}(Q)}{g_R(Q)} = \frac{C_{B-L}}{C_R}\frac{g_3(Q)}{g_2(Q)} = \tan\theta_R \tag{14.6}$$

Here it is remarkable to see that the last two equations above have structural resemblance. It is easy to show that the X_{B-L} and W_R^0 fields can mix into the X_Y gauge field over a mixing angle $\sin\theta_R$, if we were break the $SU(2)_R\times U(1)_{B-L}$ gauge symmetry into the $U(1)_Y$ symmetry through a Higgs field in the 16 Representation. This would be quite analogous with the symmetry breaking applied to the $SU(2)_L\times U(1)_Y$ symmetry where the X_Y and the W_L^0 gauge fields mix into A_μ again via a Higgs field in the 16 Representation, or as originally done with a doublet i.e., in the 2 representation. Somehow the neutral gauge fields are gradually mixed with each other to yield the massless photon. This hypothetical mixing angle $\sin\theta_R$ is analogous to $\sin\theta_L$ and is also no $mixing\ angle\ of\ SO(10)$. It will be clear in a few lines how we make use of this ratio despite the fact that it is not measured. The ratio between the coupling strength g_R of $SU(2)_R$ and the g_Y of $U(1)_Y$ is also useful. We have

$$\frac{g_R(Q)}{g_Y(Q)} = \frac{C_R}{C_Y}\frac{g_2(Q)}{g_4(Q)} = \frac{1}{\sin\theta_R} \tag{14.7}$$

This ratio can be obtained through the relations given in eq. (5.28) together with the above one. As next we consider the coupling strengths of the left and right isospin groups. We have

$$\frac{g_L(Q)}{g_R(Q)} = \frac{C_L}{C_R}\frac{g_1(Q)}{g_2(Q)} = \frac{\sin\theta_R}{\tan\theta_L} \tag{14.8}$$

Note that here $C_L = C_R = 1$. The ratio should be equal to unity at the unification mass scale M_G. This ratio can be used to define a measure of the $left-right$ asymmetry of the vacuum. For example assuming that ratio $\frac{g_L(Q)}{g_R(Q)}$ is very close to one and knowing the experimentally verified value of $\sin\theta_L$ one can determine $\sin\theta_R$ at a known mass scale. The ratio between the coupling strength g_Y of $U(1)_{B-L}$ and g_{B-L} of $U(1)_{B-L}$ is also useful. We have

$$\frac{g_{B-L}(Q)}{g_Y(Q)} = \frac{C_{B-L}}{C_Y}\frac{g_3(Q)}{g_4(Q)} = \frac{1}{\cos\theta_R} \tag{14.9}$$

This ratio can be derived from the above given relations.

In the remaining part, we will consider the so called running of the coupling strengths g_i and solve them for various mass scales embodied by $SO(10)$. The intermediate mass scales and symmetries can be determined as in Fig. (14.1). If we compare Fig. 11.1) with Fig. (14.1), one notices that they are not really equivalent. This is a technical problem. We do not know yet what values the vevs in eq. (11.3) assume therefore it is hard to guess which SSB route should be chosen in advance. We restrict ourself to expect that all the Higgs scalar in eq. (11.3) will cooperate in the SSB of $SO(10)$. Then, it would be most relevant to expect that at the scales M_G and at M_C operate the scalars Φ_{210} and Φ_{45} respectively. This was discussed in some more detail in § 11.1. As seen in Fig. (11.1) we should either proceed with $SU(3)_c\times SU(2)_R\times U(1)_{Y'}$, $SU(3)_c\times SU(2)_L\times U(1)_Y$ or with $SU(3)_c\times U(1)_{L+R}\times U(1)_{B-L}$. Phenomenologically we know that $SU(3)_c\times SU(2)_L\times U(1)_Y$ is a $dominating$ symmetry of the vacuum. Therefore it should be considered as a further intermediate symmetry with its mass scale M_W in Fig. (14.1). It is seen that along the above prescribed route all the vevs in eq. (11.3) are involved. The $\Phi_{31\bar{10}}$, Φ_{221} and Φ_{2215} Higgs scalar are nevertheless cooperating at the mass scale M_W. But this does not imply that they assume precisely equal vevs. M_W might have a fine structure in $SO(10)$. The details about this fine structure will be postponed to § 15.

$$\begin{array}{ccc}
SO(10) & & M_G \\
\downarrow & & \downarrow \\
SU(4)_c \times SU(2)_L \times SU(2)_R & & M_C \\
\downarrow & & \downarrow \\
SU(3)_c \times U(1)_{B-L} \times SU(2)_L \times SU(2)_R & & M_R \\
\downarrow & & \downarrow \\
SU(3)_c \times SU(2)_L \times U(1)_Y & & M_W \quad (+\text{ fine structure}) \\
\downarrow & & \downarrow \\
SU(3)_c \times U(1)_Q & & \text{no further SSB}
\end{array}$$

Fig. 14.1: Backbone of Descents in $SO(10)$ are on the left hand side and corresponding Intermediate mass scales are on the right hand side. A possible fine structure of M_W is suppressed.

Let us start with the running of g_Y in four separate regions. We have

$$\begin{aligned}
\frac{1}{g_4^2} &= \frac{1}{g_G^2} + 2(\frac{2}{5}b_4 + \frac{3}{5}b_2^R)\ln(\frac{M_G}{Q}), && \mathbf{M_G} > \mathbf{Q} > \mathbf{M_C} \\
\frac{1}{g_4^2} &= \frac{1}{g_C^2} + 2(\frac{2}{5}b_1^C + \frac{3}{5}b_2^R)\ln(\frac{M_C}{Q}), && \mathbf{M_C} > \mathbf{Q} > \mathbf{M_R} \\
\frac{1}{g_4^2} &= \frac{1}{g_{M_R}^2} + 2b_1\ln(\frac{M_R}{Q}), && \mathbf{M_R} > \mathbf{Q} > \mathbf{M_W}
\end{aligned} \tag{14.10}$$

The beta function for g_Y between M_C and M_R, can be derived from the following relations

$$\frac{1}{g_Y^2} = \frac{1}{g_R^2} + \frac{1}{{g_{B-L}}^2} = C_{B-L}^{-2}\frac{1}{g_3^2} + \frac{1}{g_2^2} = C_Y^{-2}\frac{1}{g_4^2} = \frac{2}{3}\frac{1}{g_3^2} + \frac{1}{g_2^2} = \frac{5}{3}\frac{1}{g_4^2} \tag{14.11}$$

from the last step follows

$$\frac{1}{g_4^2} = \frac{2}{5}\frac{1}{g_3^2} + \frac{3}{5}\frac{1}{g_2^2} \rightarrow \quad b_1 = \frac{2}{5}b_1^C + \frac{3}{5}b_2^R \tag{14.12}$$

Again the beta function for g_Y between M_G and M_C is similar to the above expression of b_1 with a replacement of b_1^C with b_4. Note that $U(1)_{B-L}$ is a subgroup of $SU(4)$. In the running of g_Y the factors $2/5$ and $3/5$ simply tell us $what$ fraction of the total renormalization is contributed to g_Y from $SU(2)_R$ and $U(1)_{B-L}$ or from $SU(4)_C$ and $U(1)_{B-L}$ interactions respectively above M_W. The running of g_L, g_{B-L}, g_R, α_s and $\alpha_{s'}$ are given collectively as

$$\begin{aligned}
g_R: \quad & \frac{1}{g_2^2} = \frac{1}{g_G^2} + 2b_2^R\ln(\frac{M_G}{Q}), && \mathbf{Q} > \mathbf{M_W} \\[2ex]
g_{B-L}: \quad & \frac{1}{g_3^2} = \frac{1}{g_G^2} + 2b_4\ln(\frac{M_G}{Q}), && \mathbf{M_G} > \mathbf{Q} > \mathbf{M_C} \\
& \frac{1}{g_3^2} = \frac{1}{g_C^2} + 2b_1^C\ln(\frac{M_C}{Q}), && \mathbf{M_C} > \mathbf{Q} > \mathbf{M_R} \\[2ex]
g_L: \quad & \frac{1}{g_1^2} = \frac{1}{g_G^2} + 2b_2^L\ln(\frac{M_G}{Q}), && \mathbf{Q} > \mathbf{M_R} \\[2ex]
\alpha_s: \quad & \frac{1}{g_5^2} = \frac{1}{g_G^2} + 2b_4\ln(\frac{M_G}{Q}), && \mathbf{M_G} > \mathbf{Q} > \mathbf{M_C} \\
& \frac{1}{g_5^2} = \frac{1}{g_C^2} + 2b_3\ln(\frac{M_C}{Q}), && \mathbf{M_C} > \mathbf{Q} \\[2ex]
\alpha_{s'}: \quad & \frac{1}{g_6^2} = \frac{1}{g_G^2} + 2b_4\ln(\frac{M_G}{Q}), && \mathbf{Q} > \mathbf{M_C}
\end{aligned} \tag{14.13}$$

The evolution of the electromagnetic coupling strength e is useful. We have

$$\begin{aligned}
\frac{3}{8}\frac{1}{e^2} &= \frac{3}{8}\frac{1}{e_G^2} + 2(\frac{2}{8}b_4 + \frac{3}{8}b_2^R + \frac{3}{8}b_2^L)\ln(\frac{M_G}{Q}), && \mathbf{M_G} > \mathbf{Q} > \mathbf{M_C} \\
\frac{3}{8}\frac{1}{e^2} &= \frac{3}{8}\frac{1}{e_C^2} + 2(\frac{2}{8}b_1^C + \frac{3}{8}b_2^R + \frac{3}{8}b_2^L)\ln(\frac{M_C}{Q}), && \mathbf{M_C} > \mathbf{Q} > \mathbf{M_R} \\
\frac{3}{8}\frac{1}{e^2} &= \frac{3}{8}\frac{1}{e_R^2} + 2(\frac{5}{8}b_1 + \frac{3}{8}b_2^L)\ln(\frac{M_R}{Q}), && \mathbf{M_R} > \mathbf{Q} > \mathbf{M_W} \\
\frac{3}{8}\frac{1}{e^2} &= \frac{3}{8}\frac{1}{e_W^2} + 2b_1^{em}\ln(\frac{M_W}{Q}), && \mathbf{M_W} > \mathbf{Q}
\end{aligned} \tag{14.14}$$

For the determination of the beta functions in the above given energy intervals, the following relations among the couplings can be used. We have

$$\begin{aligned}
\frac{3}{8}\frac{1}{e^2} = \frac{5}{8}\underbrace{(\frac{2}{5}\frac{1}{g_3^2} + \frac{3}{5}\frac{1}{g_2^2})}_{1/g_4^2} + \frac{3}{8}\frac{1}{g_1^2} &\rightarrow \frac{3}{8}\frac{1}{e^2} = \frac{2}{8}\frac{1}{g_3^2} + \frac{3}{8}\frac{1}{g_2^2} + \frac{3}{8}\frac{1}{g_1^2} \\
&\rightarrow b_1^{em} = \frac{5}{8}b_1 + \frac{3}{8}b_2^L \qquad \mathbf{M_R} > \mathbf{Q} > \mathbf{M_W} \\
&\rightarrow b_1^{em} = \frac{2}{8}b_1^C + \frac{3}{8}b_2^R + \frac{3}{8}b_2^L \qquad \mathbf{M_C} > \mathbf{Q} > \mathbf{M_R}
\end{aligned} \tag{14.15}$$

In the region $M_G < Q < M_C$, we have b_4 instead b_1^C. At mass scales $Q < M_W$ the coupling e is found as

$$\begin{aligned}
\frac{3}{8}\frac{1}{e^2(Q)} = \frac{3}{8}\frac{1}{e_G^2} &+ 2(\frac{2}{8}b_4 + \frac{3}{8}b_2^R + \frac{3}{8}b_2^L)\ln(\frac{M_G}{M_C}) + 2(\frac{2}{8}b_1^C + \frac{3}{8}b_2^R + \frac{3}{8}b_2^L)\ln(\frac{M_C}{M_R}) \\
&+ 2(\frac{5}{8}b_1 + \frac{3}{8}b_2)\ln(\frac{M_R}{M_W}) + 2b_1^{em}\ln(\frac{M_W}{Q})
\end{aligned} \tag{14.16}$$

From here e_G can be determined. But it is necessary to find the unknown scales M_G, M_C and M_R first to make use of this result. For this purpose let us summarize the couplings g_4, g_2 and g_5 by using the above set of equations for the interval $M_G < Q < M_W$. We have

$$\begin{aligned}
U(1)_Y \; &; \; \frac{1}{g_4^2(M_W)} = \frac{1}{g_G^2} + 2(\frac{2}{5}b_4 + \frac{3}{5}b_2^R)\ln(\frac{M_G}{M_C}) + 2(\frac{2}{5}b_1^C + \frac{3}{5}b_2^R)\ln(\frac{M_C}{M_R}) + 2b_1\ln(\frac{M_R}{M_W}) \\
SU(2)_L \; &; \; \frac{1}{g_1^2(M_W)} = \frac{1}{g_G^2} + 2b_2^L\ln(\frac{M_G}{M_W}) \\
SU(3)_C \; &; \; \frac{1}{g_5^2(M_W)} = \frac{1}{g_G^2} + 2b_4\ln(\frac{M_G}{M_C}) + 2b_3\ln(\frac{M_C}{M_W})
\end{aligned} \tag{14.17}$$

A possibility is to use the the combination ; $C_Y^{-2}/g_4^2 + 1/g_1^2 - (C_Y^{-2}+1)/g_5^2$. It is free of the unknown g_G and the first two terms in this combination equal to $1/e^2$. It yields;

$$\frac{1}{e^2} - \frac{8}{3}\frac{1}{g_5^2} = \mathcal{A}\ln(\frac{M_G}{M_C}) + \mathcal{B}\ln(\frac{M_C}{M_R}) + \mathcal{C}\ln(\frac{M_R}{M_W}) \tag{14.18}$$

where the constants come out as

$$\begin{aligned}
\mathcal{A} &= 2(b_2^L - 2b_4 + b_2^R) \\
\mathcal{B} &= 2(\frac{2}{3}b_1^C + b_2^R - \frac{8}{3}b_3 + b_2^L) \\
\mathcal{C} &= 2(\frac{5}{3}b_1 - \frac{8}{3}b_3 + b_2^L)
\end{aligned} \tag{14.19}$$

There are three unknowns in eq. (14.18), namely the scales M_G, M_C and M_R. Further relations are required. Another possibility comes from the weak mixing angle contained by the combination $C_Y^{-2}/g_4^2 - C_Y^{-2}/g_1^2$. We have

$$\frac{C_Y^{-2}}{g_4^2} - \frac{C_Y^{-2}}{g_1^2} = \frac{1}{e^2} - (1 + C_Y^{-2})\frac{\sin^2\theta_L}{e^2} \tag{14.20}$$

Using the expressions for g_4 and g_1 and some reorganization of the terms yield

$$\sin^2\theta_L = \frac{3}{8} - \frac{5}{8}e^2\left(\mathcal{D}\ln(\frac{M_G}{M_C}) + \mathcal{E}\ln(\frac{M_C}{M_R}) + \mathcal{F}\ln(\frac{M_R}{M_W})\right) \tag{14.21}$$

The constants $\mathcal{D}$, $\mathcal{E}$ and $\mathcal{F}$ in terms of the beta functions are found as

$$\begin{aligned}
\mathcal{D} &= 2(\frac{2}{5}b_4 + \frac{3}{5}b_2^R - b_2^L) \\
\mathcal{E} &= 2(\frac{2}{5}b_1^C + \frac{3}{5}b_2^R - b_2^L) \\
\mathcal{F} &= 2(b_1 - b_2^L)
\end{aligned} \tag{14.22}$$

Since we have three unknowns M_G, M_C and M_R in each of the statements above. It is impossible to fix them with two equations. A third one is required. Unfortunately there is no further phenomenological input other than the weak mixing angle and the coupling strengths of the electromagnetic and strong interactions. Let us impose a condition such that $M_C \lesssim M_G$. This might serve as a third relation. Formally it would be much more useful to define this inequality through a $tuning$ parameter ρ that can help with the estimation of the scales. Let

$$\rho = \ln\frac{M_G}{M_C} \geqslant 0 \tag{14.23}$$

By means of ρ, the eqs. (14.18) and (14.21) can be solved among themselves for the three unknown scales. We have

$$\begin{aligned}
M_R &= M_W \exp\left[\Delta_1 - \frac{\mathcal{B}}{\mathcal{EC}-\mathcal{FB}}\frac{1}{e^2}\left(\frac{3}{5} - \frac{8}{5}\sin^2\theta_L\right) + \frac{\mathcal{E}}{\mathcal{EC}-\mathcal{FB}}\left(\frac{1}{e^2} - \frac{8}{3}\frac{1}{g_5^2}\right)\right] \\
M_C &= M_W \exp\left[\Delta_2 + \frac{\mathcal{C}-\mathcal{B}}{\mathcal{EC}-\mathcal{FB}}\frac{1}{e^2}\left(\frac{3}{5} - \frac{8}{5}\sin^2\theta_L\right) + \frac{\mathcal{E}-\mathcal{F}}{\mathcal{EC}-\mathcal{FB}}\left(\frac{1}{e^2} - \frac{8}{3}\frac{1}{g_5^2}\right)\right] \\
M_G &= M_W \exp\left[\Delta_3 + \frac{\mathcal{C}-\mathcal{B}}{\mathcal{EC}-\mathcal{FB}}\frac{1}{e^2}\left(\frac{3}{5} - \frac{8}{5}\sin^2\theta_L\right) + \frac{\mathcal{E}-\mathcal{F}}{\mathcal{EC}-\mathcal{FB}}\left(\frac{1}{e^2} - \frac{8}{3}\frac{1}{g_5^2}\right)\right]
\end{aligned} \tag{14.24}$$

The Δ's appearing in these equations are composed of beta functions and the tuning parameter ρ. They are explicitly found as

$$\begin{aligned}
\Delta_1 &= \frac{\mathcal{DB}-\mathcal{EA}}{\mathcal{EC}-\mathcal{FB}}\rho \\
\Delta_2 &= -\frac{\mathcal{A}(\mathcal{E}-\mathcal{F}) + \mathcal{D}(\mathcal{C}-\mathcal{B})}{\mathcal{EC}-\mathcal{FB}}\rho \\
\Delta_3 &= \left(1 - \frac{\mathcal{A}(\mathcal{E}-\mathcal{F}) + \mathcal{D}(\mathcal{C}-\mathcal{B})}{\mathcal{EC}-\mathcal{FB}}\right)\rho
\end{aligned} \tag{14.25}$$

The scale factor ρ is chosen by definition positive. If we substitute the values of the beta functions, we find that

$$\Delta_1 \geqslant 0, \qquad \Delta_2 \leqslant 0, \qquad \Delta_3 \leqslant 0 \tag{14.26}$$

where Higgs scalar contributions are completely neglected in the beta functions given in eqs. (14.19) and (14.22). If ρ is allowed to increase for a fixed value of M_W, than M_G decreases and M_R increases.

14.1 Mass Scales in $SO(10)$

It should be noted that the results of the former renormalization procedure have some general validity or even a universal character as long as the contribution of the Higgs scalars are ignored. That means if the number of Higgs scalars entering the Higgs mechanism and their triplet or doublet nature is ignored, one ends up with almost the same scenario. Different symmetry breaking routes could be envisaged. But the dominant character of the intermediate electroweak symmetry does not leave to much room for alternative routes. The problem reduces mainly to the exact specification of of the Higgs multiplets initiating the same SSB's but suiting the the fermion sector in a better way. Our selection of the Higgs sector in Table (11.1) has been done in the broadest fashion where a renormalizable Yukawa sector is ensured with Majorana and Dirac masses for neutrinos. This gives the confidence to hope that the fermions masses can be reproduced by the vevs. From the other side, the Higgs sector

ρ	$\alpha_s^{-1}(M_W)$	M_G	M_C	M_R	$\alpha^{-1}(M_G)$	$\sin^2\theta_L(M_W)$	$\alpha^{-1}(M_W)$	α_G^{-1}
$\ln 1$	7	1.70×10^{19}	1.70×10^{19}	2.45×10^{9}	133.8	0.2315	128	50.2
$\ln 2$	7	8.57×10^{18}	4.28×10^{18}	9.79×10^{9}	132.9	0.2315	128	49.8
$\ln 3$	7	5.71×10^{18}	1.91×10^{18}	2.20×10^{10}	132.3	0.2315	128	49.6
$\ln 1$	8	3.09×10^{18}	3.09×10^{18}	7.67×10^{9}	131.5	0.2315	128	49.3
$\ln 2$	8	1.54×10^{18}	7.73×10^{17}	3.07×10^{10}	130.4	0.2315	128	48.9
$\ln 3$	8	1.03×10^{18}	3.43×10^{17}	6.91×10^{10}	129.9	0.2315	128	48.7
$\ln 1$	9	5.56×10^{17}	5.56×10^{17}	2.40×10^{10}	129.0	0.2315	128	48.4
$\ln 2$	9	2.78×10^{17}	1.39×10^{17}	9.62×10^{10}	128.1	0.2315	128	48.0
$\ln 3$	9	1.85×10^{17}	6.16×10^{16}	2.16×10^{11}	127.5	0.2315	128	47.8
$\ln 1$	10	1.00×10^{17}	1.00×10^{17}	7.54×10^{10}	126.6	0.2315	128	47.5
$\ln 2$	10	5.02×10^{16}	2.50×10^{16}	3.02×10^{11}	125.6	0.2315	128	47.1
$\ln 3$	10	3.34×10^{16}	1.12×10^{16}	6.78×10^{11}	125.1	0.2315	128	46.9
$\ln 1$	10	6.74×10^{16}	6.74×10^{16}	7.07×10^{10}	125.4	0.2315	127	47.0
$\ln 2$	10	3.37×10^{16}	1.68×10^{16}	2.83×10^{11}	124.1	0.2315	127	46.6
$\ln 3$	10	2.24×10^{16}	7.49×10^{15}	6.37×10^{11}	123.9	0.2315	127	46.4

Tab. 14.1: Grand unification and intermediate mass scales without contribution of any Higgs scalars. $\alpha_G = g^2/4\pi$ is the coupling strength at the grand unification mass scale M_G.

also ensures e.g. the mixing of the W_L and W_R gauge fields which leads to a richer phenomenology awaiting precision experiments.

Neglecting the contribution of the Higgs particles might not be a good approximation. Since the gauge boson masses depend very sensitively on the vevs. Let us first of all estimate the scales without any contribution of Higgs scalars using eq. (14.24). These values are summarized in Table (14.1).
It is found that M_G rests between $10^{16} - 10^{17}$ GeV and M_R between $10^{10} - 10^{11}$ GeV for various values of ρ and acceptable input values of the electroweak parameters at $M_W = 246.218$ GeV where especially $\alpha_s^{-1}(M_W) = 10$. For smaller values of α_s^{-1}, the $SO(10)$ model is no more physically viable. Because M_G moves inescapably towards the Planck scale. The various values of $\alpha(M_G)$ are also given in Table. 14.1 for the respective values of the intermediate mass scales where the fine structure constant α at M_G is obtained from eq. (14.16). We have

$$\begin{aligned}\frac{3}{8}\frac{(4\pi)^{-1}}{\alpha(Q)} &= \frac{3}{8}\frac{(4\pi)^{-1}}{\alpha(M_G)} + 2(\frac{2}{8}b_4 + \frac{3}{8}b_2^R + \frac{3}{8}b_2^L)\ln(\frac{M_G}{M_C}) + 2(\frac{2}{8}b_1^C + \frac{3}{8}b_2^R + \frac{3}{8}b_2^L)\ln(\frac{M_C}{M_R}) \\ &\quad + 2(\frac{5}{8}b_1 + \frac{3}{8}b_2)\ln(\frac{M_R}{M_W}) + 2b_1^{em}\ln(\frac{M_W}{Q})\end{aligned} \tag{14.27}$$

This expression can be solved for $\alpha(M_G)$ at $Q = M_W$. If we compare the values of α^{-1} at M_G and M_W, given in the last six rows in Table. 14.1, it is observed that they are very close. This does not happen in the effective $SU(3)_C \times SU(2)_L \times U(1)_Y$ running of α^{-1}. The difference is caused by the existence of the $U(1)_{B-L} \times SU(2)_R$ symmetries above the M_R scale. The beta functions b_1^{em} in eq. (14.15) in the interval $Q > M_R$ becomes negative. The running of α^{-1} is sketched in Fig. 14.2. As seen in the figure α^{-1} reaches a minimum value of approximately 117 at the mass scale M_R, beyond this scale the electromagnetic interactions become gradually weaker again.

14.2 Coupling Unification Beyond M_G

Towards the grand unification scale, the coupling strengths g_i converge closer and closer until they all reach the value g at M_G as was prescribed before in the preceding section. One could speculate whether these g_i's start

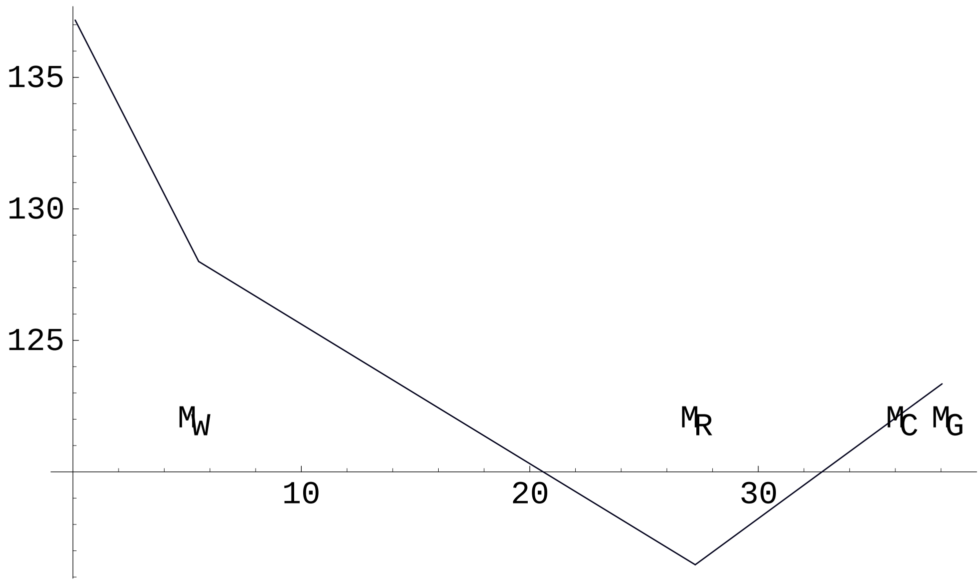

Fig. 14.2: The evolution of $1/\alpha(Q)$ (vertical axis) with respect to $\ln Q$ (horizontal axis) where Q is in GeV.

to depart from each other beyond M_G again [23]. Such a behavior would spoil unification. To illuminate this problem let us consider the case in that we run, alternatively to $SO(10)$, the standard model gauge couplings of the $SU(3)_C \times SU(2)_L \times U(1)_Y$ gauge group to some unification mass scale M_G through the following beta functions. We have

$$\begin{aligned} b_3 &= \frac{1}{(4\pi)^2}\left[-\frac{11}{3}3 + \frac{4}{3}n_g\right] \\ b_2 &= \frac{1}{(4\pi)^2}\left[-\frac{11}{3}2 + \frac{4}{3}n_g\right] \\ b_1 &= \frac{1}{(4\pi)^2}\left[-\frac{11}{3}0 + \frac{4}{3}n_g\right] \end{aligned} \tag{14.28}$$

Here it is easy to see that beyond M_G, coupling unification is no more maintained and they start to diverge, unless some Super heavy gauge bosons below M_G are introduced into the theory. These gauge bosons could contribute to the running of the couplings in such a way that the beta functions become equal. To obtain the condition $b_3 = b_2 = b_1$ the following gauge boson contributions can be considered. We have

$$\begin{aligned} b_3 &= \frac{1}{(4\pi)^2}\left[-\frac{11}{3}3 - \frac{11}{3}n_3 + \frac{4}{3}n_g\right] \\ b_2 &= \frac{1}{(4\pi)^2}\left[-\frac{11}{3}2 - \frac{11}{3}n_2 + \frac{4}{3}n_g\right] \\ b_1 &= \frac{1}{(4\pi)^2}\left[-\frac{11}{3}0 - \frac{11}{3}n_1 + \frac{4}{3}n_g\right] \end{aligned} \tag{14.29}$$

The task would be to find the appropriate multiplets of gauge bosons that transform under $SU(3)_C \times SU(2)_L \times U(1)_Y$ and produce the correct n_1, n_2 and n_3 values. An easy approach to find these bosons comes from group theory: The smallest gauge group in which the standard model can be embedded , is the rank 4 gauge group $SU(5)$, or reversely $SU(3)_C \times SU(2)_L \times U(1)_Y$ is a maximal subgroup of $SU(5)$. So the super heavy gauge bosons we are seeking are obviously those residing in the coset of $SU(5)/SU(3)_C \times SU(2)_L \times U(1)_Y$, which are the Y_α and Y'_α gauge bosons in eq. (3.26). It is found that they correctly yield $n_2 = 3$, since for each choice of color there is an $SU(2)_L$ doublet. Also $n_3 = 2$. Because for each value of weak-isospin there is one color triplet. Finally if the charges of these gauge bosons in Table (3.1) are viewed than for the same hypercharge value there are 3 color and 2 weak-isospin charges possible, producing $n_1 = 5$.

In analogy we can check this for the beta functions in eq. (14.4). Beyond M_G all of the b^3, b_1^c, b_L^2 and b_R^2 functions become equal, if the $(2,2,6) \oplus (1,1,6)$ gauge bosons in the coset of the $SO(10)/\ SU(3)_C \times U(1)_{B-L} \times SU(2)_L \times SU(2)_R$ are considered. These super heavy gauge bosons will be massless above the M_G scale and will contribute to the beta functions, as in the former case, so that coupling unification is assured. The main reason that we highlighted this discussion is due to a simple fact: $SO(10)$ has no isomorphism with any unitary group $SU(N)$ that possesses exactly 45 gauge bosons. Consequently, coupling unification above M_G is less obvious and requires some analysis.

15. THE VACUUM EXPECTATION VALUES : A NUMERICAL FIT

In this part, we will attempt to estimate various quantities that we derived from our $SO(10)$ model. As we have pointed out previously, the numerical estimates of the following quantities ;

1.1 Gauge boson masses $(W_1^{\pm}, Z_1)$, $(W_2^{\pm}, Z_2)$, $(W_3, \ldots, W_7)$.
1.2 Fermion masses of the third family : $m_t, m_b, m_{\nu_{\tau_1}}, m_{\nu_{\tau_2}}, m_\tau$
1.3 Mixing parameters : $\beta_1, \beta_2, \beta_3, \beta_1', \beta_2', \beta_3'$ and $\xi_1, \xi_2, \xi_3, \xi_4$
1.4 Phases for CP violation : $\zeta_1, \zeta_2, \zeta_3, \zeta_4$

require the following as input:

2.1 The values of the vacuum expectation values $(k, u, v, v_L, v_R, x, z)$
2.2 The values of the phases : $\gamma, \beta, \alpha, \delta$ and θ
2.3 The values of the coupling strengths g, g_L, g_R and g_{B-L} at relevant mass scales

Since we do not know the values of the Higgs coupling at any scale, it is not possible to evaluate the vevs from the minimum of the Higgs potential in § 11.2.1 despite of the fact that we have successfully solved the minimum for each of the vevs separately. Therefore the $SO(10)$ model looses its predictive power to some extent. Actually the situation is indeed not so hopeless. We have found from the minima of the Higgs potential that the vacuum looses its invariance under the $SU(2)_L$ and $SU(2)_R$ symmetries in an hyperbolic fashion; this is manifest through the two expressions in eq. (11.12). We recall the latter one which reads

$$v_L\, v_R = c_1\, k^2 + c_2\, u^2 \tag{15.1}$$

Herein we have replaced the phases and the Higgs couplings with c_1 and c_2 so that the expression looks simpler. Remarkably there exist a further relation which resembles the above dependence of $v_L\, v_R$ on k and u. This relation comes from the τ-neutrino mass $m_{\nu_{\tau_1}}$ which was given in eq. (12.11). We know that either $m_{\nu_{\tau_1}}$ or $m_{\nu_{\tau_2}}$ should have a nearly vanishing mass. Because one of them should be identified with the observed left-handed neutrino and the other with the hypothetical heavier right-handed neutrino. If we set the expression for the τ-neutrino mass identically to zero and solve it subsequently for $v_L v_R$ then we obtain exactly

$$32\, v_L v_R = k^2 + 12\, u^2 + 4\,\sqrt{3}\, k\, u\, \cos(\alpha - \delta) \tag{15.2}$$

It is seen that the last two equations above are similar up to an interference term between k and u. We will return to this point later. Let us continue our analysis with the fermions that have Dirac masses. If we look at the expressions for m_t, m_b and m_τ in eqs. (12.5) and (12.6), we see that they depend on:

3.1 the vacuum expectation values k, u and v
3.2 the phases α, δ and θ

Consequently there are two predictions that we can infer from the expressions of the quark and charged lepton masses in eqs. (12.5) and (12.6) :

4.1 Since the quark masses lie all below the Fermi scale i.e., $G \approx 246$ GeV and even below the top quark mass i.e., 174 GeV, we expect k, u and v to be approximately at the order of the Fermi scale. i.e., $\mathcal{O}(2)$.
4.2 Since the top quark mass is bigger than the bottom quark mass, we expect u to be larger than v and k to assume a median value. With this assumptions, we can account for the big mass gap between the bottom quark and the top quark.

From the other side in § 14, we had estimated the mass scales in $SO(10)$. We had found that $M_R \approx 10^{10.5\pm0.5}$ GeV as summarized shown in Table. (14.1). From the remark stated above in step (4.1) and from the expressions in eqs. (15.1) and (15.2), we arrive at the following relation:

$$v_L \cdot \underbrace{10^{10.5\pm0.5}}_{v_R} \cong G^2 \qquad \rightarrow \qquad v_L \cong 0 \tag{15.3}$$

This suggests that v_L is an extremely small number roughly equal to $6 \cdot 10^{-5.5\pm0.5}$ GeV and can be practically taken as zero in all evaluations. The above statement is extremely powerful. We can approximate the expression of the gauge boson masses, mixing parameters and phases with respect to the condition; $v_R \gg v_L \cong 0$. In this respect, let us consider the mass of the $W_1^{\pm}$ boson in eq. (11.29). The mass of $W_1^{\pm}$ should be equal to the mass of the $W^{\pm}$ boson of the electroweak theory. i.e., 80 GeV. If we approximate the expression for $M_{W_1^{\pm}}$ in eq. (11.29) according to $v_R \gg v_L \cong 0$, we obtain

$$(M_{W_1^{\pm}})^2 \simeq \alpha_6 \cong \frac{1}{4}\, g_L^2\, \left(k^2 + u^2 + v^2 + 2\, v_L^2\right) \cong (80)^2\text{GeV}^2 \rightarrow \left(k^2 + u^2 + v^2\right) \cong (246)^2\text{GeV}^2 \tag{15.4}$$

Since $v_L \cong 0$, we end up with the equality on the right hand side above. Consequently we have recovered the remark in step (4.1) which tells us that the relation in eq.(15.3) holds consistently. But unfortunately, it doesn't reveal us the values of k, u and v separately. A straight forward approach is to seek those k, u and v values that can reproduce the fermion masses and simultaneously satisfy the condition:

$$\left(k^2 + u^2 + v^2\right) \cong (246)^2\text{GeV}^2. \tag{15.5}$$

With this condition, we will be able to unveil the input parameters of $SO(10)$ grand unification. The values of the k, u and v vevs that can reproduce the fermion masses will be studied as next.

15.1 Fermion Masses

In the guidance of the remark in point (4.2) above, it becomes suitable to substitute the values $u = 246 \pm 40$ GeV, $v = 1$ GeV, $k = 4$ GeV, $\cos(\alpha - \delta) = -1$ and $\cos(\alpha + \theta) = 1$ into the exact expressions for the top and bottom quark masses in eq. (12.5). These input values are summarized in Table (15.2), We obtain

$$\begin{aligned} m_t &= 51 \pm 8 \;\text{ GeV} \\ m_b &= 0.91 \;\text{ GeV} \end{aligned} \tag{15.6}$$

These are the top and bottom quark masses at the grand unification mass scale $M_G = 3.37 \cdot 10^{16}$ GeV where all the entries of the Yukawa coupling are equal to 1 as in eq. (12.3). These quark masses should be renormalized to energies below the Fermi scale by using the QCD renormalization group equations [99][100][101]. The top quark mass renormalizes to $m_t(m_t) \approx 142 \pm 25$ GeV and the bottom quark mass renormalizes to $m_b(m_b) \approx 4$ GeV. Furthermore we substitute the same same input values in Table (15.2) into the exact expressions for the τ-lepton and τ-neutrino masses given in eq. (12.6). We obtain

$$\begin{aligned} m^D_{\nu_\tau} &= 150.41 \pm 25 \;\text{ GeV} \\ m_\tau &= 0.76 \;\text{ GeV} \end{aligned} \tag{15.7}$$

These are the Dirac masses of the leptons at the grand unification mass scale M_G where again all the entries of the Yukawa coupling are equal to 1 as in eq. (12.3). The τ-lepton mass m_τ roughly renormalizes down to 1800 MeV at 2 GeV. Let us continue with the Majorana mass terms of the τ-neutrino given in eq. (12.7). Using the same input values in Table (15.2), we obtain

$$\begin{aligned} m^M_{\nu_{\tau_L}} &\cong 0 \\ m^M_{\nu_{\tau_R}} &= 10^{10} \;\text{ GeV} \end{aligned} \tag{15.8}$$

τ**-neutrino Mass in** $SO(10)$ **:** $m_{\nu_{\tau_1}}$ **and** $m_{\nu_{\tau_2}}$			
v_L (GeV)	v_R (GeV)	$m_{\nu_{\tau_1}}$(eV)	$m_{\nu_{\tau_2}}$ (GeV)
$6 \cdot 10^{-6}$	10^{10}	$5{,}72^{+1.88}_{-1.90}$	10^{10}
$6 \cdot 10^{-5}$	10^{9}	$75{,}10^{+13.47}_{-16.10}$	10^{9}
$6 \cdot 10^{-4}$	10^{8}	$750{,}36^{+134.91}_{-158.89}$	10^{8}
$6 \cdot 10^{-3}$	10^{7}	$7503{,}81^{+1349.07}_{-1589.07}$	10^{7}
$6 \cdot 10^{-2}$	10^{6}	$750038{,}10^{+13490.7}_{-15890.7}$	10^{6}

Tab. 15.1: τ-neutrino mass in $SO(10)$ at the grand unification mass scale M_G for various values of v_R and v_L where $v_L v_R = G^2$ and G equals the Fermi scale.

Herein we have taken $v_R = 10^{10}$ GeV and $v_L \cong 0$. The values of the phases β and γ satisfy the condition $e^{i(\beta-\gamma)/2} = 1$. These were absorbed in the Majorana-neutrino flavor-eigenbasis in eq. (12.9). The above masses are therefore absolute values. Since we want to evaluate the τ-neutrino masses $m_{\nu_{\tau_1}}$ and $m_{\nu_{\tau_2}}$, we may proceed in two ways: We can either find the mass eigenvalues of the mass matrix $\mathcal{M}$ in eq. (12.8) whose entries are $m^M_{\nu_{\tau_L}}$, $m^M_{\nu_{\tau_R}}$ and $m^D_{\nu_\tau}$ which were evaluated above or we can directly substitute the input values into the exact expressions of $m_{\nu_{\tau_1}}$ and $m_{\nu_{\tau_1}}$ in eq. (12.11). In both cases we obtain the same result. We have

$$
\begin{aligned}
m_{\nu_{\tau_1}} &= 5.72^{+1.88}_{-1.90} \ \text{eV} \\
m_{\nu_{\tau_2}} &= 2 \cdot 10^{10} \ \text{GeV}
\end{aligned}
\tag{15.9}
$$

It is very remarkable that the above value of $m_{\nu_{\tau_1}}$ falls into the eV range. Note that we have taken v_L exactly zero. If we repeat the same evaluation with $v_L = 6 \cdot 10^{-6}$ GeV which was estimated from eq. (15.3), the mass $m_{\nu_{\tau_1}}$ does not suffer any change. Before we close this section, we will continue to elaborate the τ-neutrino mass $m_{\nu_{\tau_1}}$ by using different values of v_L and v_R that satisfy the condition:

$$
v_L \, v_R = G^2 \tag{15.10}
$$

The values of k, u, v and the phases δ, θ, α will be again like those given in Table (15.2). As seen in Table (15.1), for $v_L = 6 \cdot 10^{-5}$ GeV, $m_{\nu_{\tau_1}}$ enhances to $\sim$ 75 eV which corresponds to $v_R = 10^9$ GeV. As we decrease v_R each time by a factor of 10 and thereby increase v_L by a factor of 10, $m_{\nu_{\tau_1}}$ increases by a factor of 10 as seen in Table (15.1).

These small values of v_L do not violate the condition in eq. (15.5). That means our model will still predict the $W^\pm$ boson mass of the electroweak theory correctly hence with a surplus of heavier τ-neutrino masses. The question we have to ask ourself is how small v_R could be in $SO(10)$? As shown in § 14 where we mainly studied the coupling unification in $SO(10)$, we found that v_R rests at 10^{10} GeV. In those evaluations we neglected the contribution of scalar bosons in the beta functions. We may expect that the contribution of Higgs scalars reduce M_R maximally by two orders of magnitude. Then the τ-neutrino mass $m_{\nu_{\tau_1}}$ might be only as big as $\sim$ 750 eV at the grand unification mass scale as shown in Table (15.1). We find the τ-neutrino masses in the first two rows in Table (15.1) as favorable. Last but not least, our $SO(10)$ model predicts non-vanishing neutrino masses which are consistent with the current status of neutrino physics. A summary of the quark and lepton masses obtained

Input Values for Quark and Lepton Masses in $SO(10)$						
u	v	k	v_L	v_R	$\cos(\alpha+\theta)$	$\cos(\alpha-\delta)$
246 ± 40	1	4	$6 \cdot 10^{-6}$	10^{10}	1	-1

Tab. 15.2: Input values for quark and lepton masses in $SO(10)$. All values of the vevs are in GeV.

Quark and Lepton Masses in $SO(10)$				
Quarks		Leptons		
m_t	m_b	m_τ	$m_{\tau_{\nu_1}}$	$m_{\tau_{\nu_2}}$
51 ± 8 GeV	0.91 GeV	760 MeV	$5.72^{+1.88}_{-1.90}$ eV	10^{10} GeV
142 ± 25 GeV	4 GeV	~ 1800 MeV	~ 14 eV	–

Tab. 15.3: Quark and lepton masses of the 3^{rd} generation in $SO(10)$. In the first row, masses are given at the grand unification mass scale. In the second row, quark masses are given like $m_t(m_t)$ and $m_b(m_b)$. Lepton masses are given at 2 GeV.

for the heaviest fermion generation is given in Table (15.3). The evaluation of the fermion masses of the first and second generation requires a renormalization procedure of the Yukawa couplings which we won't undertake. We are optimistic that such a renormalization procedure can reproduce the remaining fermion masses in the realm of $SO(10)$ grand unification.

15.2 Gauge Boson Masses

In this section, we will evaluate the masses of the physical gauge bosons in $SO(10)$ which were derived from the mass-squared matrix in § 11.3. The expressions for the physical gauge boson masses were previously given in eq. (11.3.2). Using the input values in Table (15.4), we obtain

$$\begin{aligned} M_A &= 0 \\ M_{W_1^\pm} &\cong 80.10^{-6.46}_{+6.60} \quad \text{GeV} \\ M_{Z_1} &\cong 91.21^{-7.30}_{+7.09} \quad \text{GeV} \end{aligned} \tag{15.11}$$

Input Values for Gauge Boson Masses in $SO(10)$								
u	v	k	v_L	v_R	x	z	$\cos(\alpha+\theta)$	$\cos(\alpha-\delta)$
246 ± 40	1	4	$6 \cdot 10^{-6}$	10^{10}	$\frac{3.38 \cdot 10^{16}}{e^2}$	$3.38 \cdot 10^{16}$	1	-1
$\frac{g_L^2(G)}{4\pi}$	$\frac{g_R^2(G)}{4\pi}$	$\frac{g_{B-L}^2(G)}{4\pi}$	$\frac{g_L^2(M_R)}{4\pi}$	$\frac{g_R^2(M_R)}{4\pi}$	$\frac{g_{B-L}^2(M_R)}{4\pi}$	$\frac{g^2(M_G)}{4\pi}$	M_G	M_R
$\frac{1}{29.6}$	$\frac{1}{29.6}$	$\frac{1}{68.6}$	$\frac{1}{38.6}$	$\frac{1}{38.6}$	$\frac{1}{34.3}$	$\frac{1}{46.6}$	$3.38 \cdot 10^{16}$	10^{10}

Tab. 15.4: Input values for gauge boson masses in $SO(10)$. All values of the vevs are in GeV and $e \approx 2.718$ which is the euler number.

$$\begin{aligned}
M_{W_2^\pm} &\cong 4.03 \cdot 10^9 \quad \text{GeV} \\
M_{Z_2} &\cong 8.32 \cdot 10^9 \quad \text{GeV} \\
M_{W_3} &\cong 1.76 \cdot 10^{16} \quad \text{GeV} \\
M_{W_4} &\cong 1.76 \cdot 10^{16} \quad \text{GeV} \\
M_{W_5} &\cong 1.76 \cdot 10^{16} \quad \text{GeV} \\
M_{W_6} &\cong 2.74 \cdot 10^{15} \quad \text{GeV} \\
M_{W_7} &\cong 1.76 \cdot 10^{16} \quad \text{GeV}
\end{aligned} \tag{15.12}$$

Herein we have used the same input values that we used to evaluate quark masses. The errors in the masses of $W_1^\pm$ and Z_1 stem form the vev u. Additionally required input values for the above evaluations are summarized in Table (15.4). For example the evaluation of the $W_1^\pm$ and Z_1 gauge boson masses require the values of the coupling strengths g_L, g_R and g_{B-L} at the mass scale G as shown in Table (15.4). The grand unification mass scale has been chosen as $M_G = 3.38 \cdot 10^{16}$ GeV. The corresponding grand unification coupling strength is $\alpha_G = g^2/4\pi = 1/46.6$ which was obtained in § 14. For the tuning parameter, we have $\rho = \ln[M_G/M_C] = 2$. Note that smaller values of ρ may be acceptable as well but this has physical consequences which will be discussed later in § 15.3. From the other side the evaluation of the masses of the $W_2^\pm$ and Z_2 gauge bosons require the values of the coupling strengths of g_L, g_R and g_{B-L} at the mass scale $M_R = 10^{10}$ GeV. These are also summarized in Table (15.4). They are obtained from the renormalization equations of the coupling strengths given in eq. (14.13). As we compare the masses of the W_6 gauge bosons with the other $W3, W_4, W_5, W_7$ gauge bosons, we see that it is roughly 6 to 7 times lighter. W_6 and W_7 mediate $B-L$ violating processes. The lightness of W_6 will have interesting consequences. A lower mass will increase the rate of $B-L$ violating processes and will thereby reduce the life-time of various decays. The lightness of W_6 is a direct consequence of $\rho = 2$.

15.3 Mixing Parameters and Phases

15.3.1 Mixing Parameters of the Charged and Colored Fields

In this section, we will evaluate the mixing parameters ξ_1, ξ_2 and ξ_3 which resulted from the mass-squared matrix in § 11.3. Using the input values in Table (15.4), we obtain

$$\text{Charged currents} \quad \{ \ \xi_1 \cong 0.02631$$

$$\text{Charged + colored currents} \quad \begin{cases} \xi_2 \cong \ 0.714^{-0.08}_{+0.09} \\ \\ \xi_3 \cong \ \begin{cases} \frac{\pi}{4} & \text{if } x = z \\ 0 & \text{if } x \neq z \end{cases} \end{cases} \tag{15.13}$$

Here we find it appropriate to address the following two cases related with the value of ξ_1:

- Case A: If the vacuum expectation values v_L and v_R *were* equal then the gauge fields $W_L^{\pm}$ and $W_R^{\pm}$ would mix through the angle $\frac{\pi}{4}$ radians. In this special case the determination of the fermion masses and gauge bosons masses becomes in our opinion impossible. This picture obviously does not suit the pattern in nature. We will not speculate about the reasons. Our evaluations predict $v_R = 10^{10}$ Gev $\gg v_L \approx 0$ GeV.
- Case B: The other case is when v_R is much greater than v_L then $W_L^{\pm}$ and $W_R^{\pm}$ start to decouple. At $M_R = 10^{10}$ GeV, the mixing angle ξ_3 assumes the above value. Beyond this value at roughly $M_R > 2.5 \cdot 10^{10}$ GeV they are completely decoupled. Stringent experimental bounds on ξ_3 as reviewed in ref. [24] can allow us to estimate a lowest bound on M_R.

As seen above, the value for ξ_2 is very close to $\frac{\pi}{4}$ and tells us that A'_α and Y_α mix almost in one to one proportion. From the other side there are two cases of interest for ξ_3:

- Case A': If the grand unification mass scales M_G equals the intermediate mass scale M_C, then the gauge fields A_α and X_α mix through the angle $\frac{\pi}{4}$. In this case the masses of the W_3, W_4, W_5, W_6, W_7 gauge bosons become equal and increase by a factor of 1.14 with respect to the values given in eq. (15.11). Such a picture can be physically viable as seen in Table (14.1).
- Case B': If M_C is smaller than M_G then the gauge fields A_α and X_α will start to decouple. At a difference of 100 GeV the mixing angle takes the value $\xi_3 = 1.6 \cdot 10^{-6}$ radians. The mass of the W_6 boson becomes gradually smaller than the W_3, W_4, W_5, W_7 bosons depending on the ratio $\rho = M_G/M_C$. Such a picture is also physically viable as seen in Table (14.1) and reproduces our model for $\rho = 2$.

It is remarkable how a small difference between the scales M_C and M_G can be so decisive on ξ_3 in W_6 mass. The decoupling of A and X means that $B - L$ violating processes become extremely less probable. In fact 100 GeV compared to the grand unification mass scale can suitably be considered as a fluctuation. At $kT \approx 3.38 \cdot 10^{16}$ GeV the grand unification mass scale and beyond of it can be considered as a condensate of quarks and leptons in equilibrium with gauge boson and Higgs bosons . As the universe cools down roughly 100 GeV down A and X will almost be decoupled with $\xi_3 = 1.6 \cdot 10^{-6}$. Such processes will contribute to the net excess of baryons over anti-baryons as the universe cools down.

15.3.2 Mixing Parameters of the Neutral Fields: Vector and Axial-vector Couplings in $SO(10)$

The mixing parameters appearing in the expression for Z_1 in eq. (11.20) can be evaluated at the mass scale G using the input values in Table (15.4). We have

$$\begin{aligned} \beta_1 &\cong -2.1792 & &\rightarrow & \frac{\beta_1}{\mathcal{N}_1} &\cong -0.8765 \\ \beta_2 &\cong \ 0.6568 & &\rightarrow & \frac{\beta_2}{\mathcal{N}_1} &\cong \ \ 0.2642 \\ \beta_3 &= \ 1 & &\rightarrow & \frac{\beta_3}{\mathcal{N}_1} &\cong \ \ 0.4022 \end{aligned} \tag{15.14}$$

where the normalization constant $1/\mathcal{N} \approx 0.4022$. The mixing parameters appearing in the expression for Z_2 in eq. (11.21) can be evaluated at the mass scale M_R using the input values in Table (15.4). We have

$$\begin{aligned}
\beta_1' &\cong 0 && \rightarrow && \frac{\beta_1'}{\mathcal{N}_1'} \cong 0 \\
\beta_2' &\cong 0.9428 && \rightarrow && \frac{\beta_1'}{\mathcal{N}_1'} \cong -0.6859 \\
\beta_3' &= 1 && \rightarrow && \frac{\beta_1'}{\mathcal{N}_1'} \cong 0.7276
\end{aligned} \tag{15.15}$$

where the normalization constant $1/\mathcal{N}' \approx 0.7276$. The above mixing parameters $\beta_1, \beta_2, \beta_3$ and $\beta_1', \beta_2', \beta_3'$ are governing the mixing among neutral fields W_L, W_R and X_{B-L} which we studied in § 13.2 and thereby they are related with the previously defined vector and axial-vector couplings in eq. (13.6). We find it appropriate to study the vector and axial-vector couplings in terms of the above evaluations of $\beta_1, \beta_2, \beta_3$ and $\beta_1', \beta_2', \beta_3'$. But first of all these mixing parameters must be evaluated at the same mass scale, preferentially at the Fermi scale G before we collect them in the matrix in eq. (13.4). Using the input values in Table (15.4), we find

$$\begin{bmatrix} A \\ Z_1 \\ Z_2 \end{bmatrix} \approx \begin{bmatrix} 0.4812 & 0.4812 & 0.7326 \\ -0.8765 & 0.2642 & 0.4022 \\ 0 & -0.8358 & 0.5490 \end{bmatrix} \begin{bmatrix} W_L^0 \\ W_R^0 \\ X_{B-L} \end{bmatrix} \tag{15.16}$$

Note that all rows in the matrices are normalized properly to 1. The entries of the second row is as in eq. (15.14). The third rows is re-evaluated at the mass scale G. We need the inverse of this matrix to substitute the W_L, W_R and X_{B-L} fields in term of the A, Z_1 and Z_2 fields into the interaction Lagrangian of the neutral currents in eq. (13.5). The inverse matrix reads

$$\begin{bmatrix} W_L^0 \\ W_R^0 \\ X_{B-L} \end{bmatrix} \approx \underbrace{\begin{bmatrix} 0.4812 & -0.8765 & 0 \\ 0.4812 & 0.2642 & -0.8358 \\ 0.7326 & 0.4022 & 0.5490 \end{bmatrix}}_{m_{ij}} \begin{bmatrix} A \\ Z_1 \\ Z_2 \end{bmatrix} \tag{15.17}$$

The entries of this inverse matrix will be called m_{ij} with $i, j = (1, 2, 3)$. If we substitute the gauge fields W_L, W_R and X_{B-L} into the interaction Lagrangian of the neutral currents in eq. (13.5), we obtain

$$\begin{aligned}
\mathcal{L} &= +i \left[g_L \, \bar{f} \, \gamma_\mu \, \frac{1}{2}(1-\gamma_5) \, f \, L_3 \, W_L^0 + g_R \, \bar{f} \, \gamma_\mu \, \frac{1}{2}(1+\gamma_5) \, f \, R_3 \, W_R^0 + \sqrt{\frac{2}{3}} \, g_{B-L} \bar{f} \gamma_\mu \, f \, (B-L) \, X_{B-L} + h.c. \right] \\
&= +i \left[g_L \, \bar{f} \, \gamma_\mu \, \frac{1}{2}(1-\gamma_5) \, f \, L_3 \, (m_{11} \, A + m_{12} \, Z_1 + m_{13} \, Z_2) + g_R \, \bar{f} \, \gamma_\mu \, \frac{1}{2}(1+\gamma_5) \, f \, R_3 \, (m_{21} \, A + m_{22} \, Z_1 \right. \\
&\qquad \left. + m_{23} \, Z_2) + \frac{1}{2} \, g_{B-L} \bar{f} \, \gamma_\mu \, f \, X_{B-L} \, (B-L) \, (m_{31} \, A + m_{32} \, Z_1 + m_{33} \, Z_2) + h.c. \right]
\end{aligned} \tag{15.18}$$

Herein f shortly denotes fermions. Note that the neutral isospin currents are projected through the projection operators P_L and P_R into J_L^0 and J_R^0 respectively where $P_{L,R} = (1 \mp \gamma_5)/2$. The currents J_{B-L}, J_L^0 and J_R^0 in $SO(10)$ were given previously in eqs. (5.13) and (5.21) respectively. Note that the factor $\sqrt{3/2}$ in the first line above cancels out with the factor $\sqrt{2/3}$ which comes from the expression for the J_{B-L} current. We also receive a factor of $1/2$ from the J_{B-L} current which properly accounts for the charge relation $2Q = L_3 + R_3 + (B-L)$ where L_3, R_3 and $(B-L)$ above are the eigenvalue operators. The above Lagrangian $\mathcal{L}$ can be further organized into a more useful form. We have

$$\begin{aligned}
\mathcal{L} = +i \Bigg\{ \bar{f} \, \gamma_\mu \, \Bigg(& \left[\frac{1}{2} \left(g_L \, m_{11} \, L_3 + g_R \, m_{21} \, R_3 + g_{B-L} \, m_{31} \, (B-L) \right) + \frac{1}{2} \left(g_R \, m_{21} L_3 - g_L \, m_{11} R_3 \right) \, \gamma_5 \right] \, A^\mu \\
& + \left[\frac{1}{2} \left(g_L \, m_{12} \, L_3 + g_R \, m_{22} \, R_3 + g_{B-L} \, m_{32} \, (B-L) \right) - \frac{1}{2} \left(g_L \, m_{12} \, L_3 - g_R \, m_{22} \, R_3 \right) \, \gamma_5 \right] \, Z_1^\mu \\
& + \left[\frac{1}{2} \left(g_L \, m_{13} \, L_3 + g_R \, m_{23} \, R_3 + g_{B-L} \, m_{33} \, (B-L) \right) - \frac{1}{2} \left(g_L \, m_{13} \, L_3 - g_R \, m_{23} \, R_3 \right) \, \gamma_5 \right] \, Z_2^\mu \Bigg) + h.c. \Bigg\} \, f
\end{aligned} \tag{15.19}$$

Vector and Axial-vector Couplings of NC Currents in $SO(10)$:								
	Q	B-L	I_{3R}	I_{3L}	c_V^f	c_A^f	$c_{V'}^f$	$c_{A'}^f$
ν_e, ν_μ, ν_τ	0	-1	$\frac{1}{2}$	$\frac{1}{2}$	-0.371651	-0.371654	-0.507273	0.272296
e, μ, τ	-1	-1	$-\frac{1}{2}$	$-\frac{1}{2}$	0.027337	0.371654	0.037319	-0.272296
u, c, t	$\frac{2}{3}$	$\frac{1}{3}$	$\frac{1}{2}$	$\frac{1}{2}$	-0.142108	-0.371654	-0.193971	0.272296
d, s, b	$-\frac{1}{3}$	$\frac{1}{3}$	$-\frac{1}{2}$	$-\frac{1}{2}$	0.256879	0.371654	0.350622	-0.272296

Tab. 15.5: Vector and axial-vector couplings of NC currents in $SO(10)$. c_V^f and c_A^f are the couplings for Z_1. $c_{V'}^f$ and $c_{A'}^f$ are the couplings for Z_2 as defined in eq. (13.6).

The above Lagrangian is identical with the one in eq. (13.6). Let us examine the first row which is nothing but the electromagnetic interaction Lagrangian $\mathcal{L}_{em}$. We should have

$$(i) \quad e = g_L\, m_{11} = g_R\, m_{21} = g_{B-L}\, m_{31} \qquad \text{and} \qquad (ii) \quad g_R\, m_{21} L_3 - g_L\, m_{11} R_3 = 0 \tag{15.20}$$

The above equalities in (i) and (ii) should hold regardless of their numerical values. They are algebraically always true since the $U(1)_Q$ symmetry is unbroken. Nevertheless with respect to our input values in Table (15.4), we find that the first equalities in (i) hold with an error of one part in 10^{-5} and the latter equality in (ii) holds with an error of 4 parts in 10^{-6}. Both conditions hold very accurately. Consequently we can factor out e so that the sum of the L_3, R_3 and $(B-L)$ eigenvalue operators make the electric-charge eigenvalue operator $2\,Q$. Thereby the first line reproduces the electromagnetic interaction Lagrangian. From the other side the matrix elements in the second and third line above can be set equal to the previously defined vector and axial-vector couplings in eq. (13.6). We have

$$\begin{aligned} c_V^f &= g_L\, m_{12}\, L_3 + g_R\, m_{22}\, R_3 + g_{B-L}\, m_{32}\, (B-L) \\ c_A^f &= g_L\, m_{12} R_3 - g_R\, m_{22} L_3 \\ \\ c_{V'}^f &= g_L\, m_{13}\, L_3 + g_R\, m_{23}\, R_3 + g_{B-L}\, m_{33}\, (B-L) \\ c_{A'}^f &= g_L\, m_{13}\, L_3 - g_R\, m_{23}\, R_3 \end{aligned} \tag{15.21}$$

These equations require some special care during any evaluation. Since we are used to assign fermions the quantum numbers as in the standard model, we remind the reader: For example if f is an up-quark (u), it will posses the charges $L_3 = 1/2$, $R_3 = 1/2$ and $B-L = 1/3$. If f is a down-quark (d), it will posses the charges $L_3 = -1/2$, $R_3 = -1/2$ and $B-L = 1/3$. On the other side if f is an up-lepton (ν), it will possess the charges $L_3 = 1/2$, $R_3 = 1/2$ and $B-L = -1$. If f is a down-lepton (e), it will posses the charges $L_3 = 1/2$, $R_3 = 1/2$ and $B-L = -1$. Note that the electric charges of fermions are obtained trough $(1/2)(L_3 + R_3 + (B-L))$ since we deal with fermions (f) and not with chiral states f_L or f_R (see remark at the end of this section). With this token, we can evaluate the above expressions of $c_V^f, c_A^f, c_{V'}^f$, and $c_{A'}^f$ using the values of m_{ij} in eq. (15.17) and the values of the coupling strengths g_L, g_R and g_{B-L} at the mass scale G as given in Table (15.4). Note that the index i in m_{ij} runs along the row elements and j along the column elements. The values of these couplings are summarized in the Table. (15.5). At this stage we can ask ourselves whether the above values of the vector and axial-vector couplings of Z_1 in $SO(10)$ are the same with those of the Z boson of the electroweak theory. Let us call the latter couplings as C_V^f and C_V^f with a capital C to avoid any confusion in advance. Indeed we have to remember that the vertex for NC currents in the electroweak theory is given through

$$-\,i\frac{g}{\cos\theta_w}\gamma_\mu \frac{1}{2}\left(C_V^f - C_A^f\,\gamma_5\right) \tag{15.22}$$

	Q	B-L	I_{3R}	I_{3L}	c_V^f	c_A^f	$c_{V'}^f$	$c_{A'}^f$
ν_e, ν_μ, ν_τ	0	-1	$\frac{1}{2}$	$\frac{1}{2}$	0.5000040	0.500004	0.682458	-0.366333
e, μ, τ	-1	-1	$-\frac{1}{2}$	$-\frac{1}{2}$	-0.0367777	-0.500004	-0.050207	0.366333
u, c, t	$\frac{2}{3}$	$\frac{1}{3}$	$\frac{1}{2}$	$\frac{1}{2}$	0.1911852	0.500004	0.260958	-0.366333
d, s, b	$-\frac{1}{3}$	$\frac{1}{3}$	$-\frac{1}{2}$	$-\frac{1}{2}$	-0.3455928	-0.500004	-0.471708	0.366333

Vector and Axial-vector Couplings of NC Currents in $SO(10)$
A Comparison with the Electroweak Theory

Tab. 15.6: Vector and axial-vector couplings of NC currents in $SO(10)$ divided by a factor of $\frac{-g}{\cos\theta_w}$ appearing as vertex factor in the electroweak theory. See explanation on page 109.

where g is the *weak coupling constant* and θ_w is the Weinberg mixing angle and C_V^f and C_A^f are the vector and axial-vector couplings non of which should be confused with those of the $SO(10)$ model. Now to make it easy to compare the above c_V^f and c_A^f couplings of the $SO(10)$ model evaluated from the expressions in eq. (15.21) with those of the electroweak theory, we will divide the values in Table (15.5) through the factor

$$\frac{-g}{\cos\theta_w} \approx -0.743302 \tag{15.23}$$

These values are summarized in Table. (15.6). We see from Table (15.6) that the vector and axial-vector couplings of the Z boson in the electroweak theory and those of the Z_1 boson in the $SO(10)$ theory are in good agreement with respect to our evaluations. From the other side the Z_2 boson would be to heavy to be produced in current accelerators. Therefore any mismatch between the vector and axial-vector couplings of the $SO(10)$ theory and the electroweak theory could be searched in precision experiments as an indication for physics beyond the standard model.

Remark : In case that we use fermion fields which are handed like f_L or f_R, the electric-charge relation reads

$$Q = L_3 + R_3 + \frac{(B-L)}{2} \tag{15.24}$$

But in our interaction Lagrangian the currents couple to f and not to f_L or to f_R. Therefore the electric-charge relation reads

$$Q = \frac{L_3 + R_3 + (B-L)}{2} \tag{15.25}$$

which can be verified to hold. Let us give an example: u_L has $L_3 = 1/2$, $R_3 = 0$ and $B - L = 1/3$. And u has $L_3 = 1/2$, $R_3 = 1/2$ and $B - L = 1/3$. Both yield $Q = 2/3$. This is also dictated to us by the interaction Lagrangian in eq. (15.19). Note that the factor $1/2$ in the first term in eq. (15.19) correctly multiplies $2\,Q$ and reproduces the correct electromagnetic interaction Lagrangian:

$$\mathcal{L}^{em} = +\,i\left\{e\,\left(\bar{f}\gamma_\mu\,Q\,f\right)\,A^\mu + h.c.\right\} \tag{15.26}$$

15.3.3 *The CP-Phases of the Charged and Colored Fields in $SO(10)$*

In this section, we will evaluate the phases ζ_1, ζ_2 and ζ_3 which resulted from the mass-squared matrix in § 11.3. The expression of these phases were given in eqs. (11.26) and (11.27). ζ_1 one appeared in the expression of $W_1^\pm$

and $W_2^{\pm}$ in eqs. (11.20) and (11.20) and thereby entered the Lagrangian of charged currents given in eq. (13.3). It is a source for CP violation. The latter two phases appeared in the expressions of $W_4, \ldots, W_7$ in eqs. (11.22) and entered the Lagrangian of charged and colored currents given in eq. (13.12). These are also sources for CP violation. Using the input values in Table (15.4), we obtain

$$
\begin{aligned}
&\text{Charged currents}\left\{\ \zeta_1 \quad = -\frac{\pi}{4}\right. \\
&\text{Charged + colored currents}\begin{cases} \zeta_2 & = +\dfrac{\pi}{4} \\[2ex] \zeta_3 & = \arctan(-\cot\beta) = -\dfrac{\pi}{4} \end{cases}
\end{aligned}
\tag{15.27}
$$

where we have $\beta + \gamma = \pi/2$.

15.3.4 The Mixing Angle and CP-Phase of Neutrinos in $SO(10)$

We will evaluate two more parameters. These are the mixing parameter ξ_4 and the phase ζ_4 which originated from the neutrino mass matrix in eq. (12.8). They determine the complex mixing of the Majorana-neutrino flavor-eigenbasis. The expressions for ξ_4 and ζ_4 are given in eq. (12.13) and (12.14) respectively. Using again the input values in Table (15.4), we obtain

$$
\begin{aligned}
&\zeta_4 = \delta \\
&\xi_4 \cong -1.5 \cdot 10^{-7}
\end{aligned}
\tag{15.28}
$$

We have previously constrained $\cos(\alpha + \theta)$ and $\cos(\alpha - \delta)$ as in Table (15.2). Therefore the angles α, θ and δ were separately not assigned any definite value. Consequently we have $\alpha = \pi + \delta$. Through this relation ζ_4 in eq. (12.13) reduces to δ. As a result, we have here a free parameter which should be adjusted further.

It can be verified from eq. (12.12) that for the above value of ξ_4, ν_{τ_1} becomes practically left-handed and ν_{τ_2} becomes right-handed. Any experimental evidence indicating a non-zero value of ξ_4 would suggest that neutrinos have both Dirac and Majorana Masses. Of course this would require precision experiments.

16. CONCLUSION AND OUTLOOK

In this work we have studied $SO(10)$-grand unification with emphasis on fermion masses. We started our excursion by introducing the essential ingredient required for the construction of an $SO(10)$ theory of gauge interactions. In this respect, we introduced three different bases which produce different, but physically equivalent spinorial representations of the $SO(10)$ gauge group. We studied the structure of the $SO(10)$ gauge group by presenting the general form of all the fields and generators. We derived the physical charges of these gauge fields and their decompositions with respect to various sub-symmetries through certain commutation relations. We presented the explicit matrix representation of the gauge term which acts on the family spinor. We have derived the eigenvalue operators that produce the charges of the fermions, and showed that the family spinor contains all the known fermions of a single generation. Using the transformation properties of the $SO(10)$ spinorial representation, we showed that $SO(10)$ gauge interactions are C, P and CP invariant in the unbroken phase. Prior to any symmetry breakdown, we studied the interaction Lagrangian of the $SO(10)$ theory in the flavor basis, and we derived all the interaction vertices between fermions and gauge fields: in the light of this analysis, we showed that these currents conserve baryon minus lepton number, but separately violate baryon and lepton numbers at the vertices. We showed that the color carrying currents mediate various nucleon decays. Prior to any spontaneous symmetry breakdown, we showed how the electromagnetic current is formally contained by the $SO(10)$ theory. We have shown how the single gauge coupling g, defining the strength of $SO(10)$ gauge interaction is related with relative gauge couplings that are assigned to the unitary subgroups of the $SO(10)$ gauge group.

Starting from $\S$ 6 until $\S$10, we studied the Higgs fields that transform under the $SO(10)$ symmetry, in order to implement them in the Higgs mechanism. Through a detailed analysis, we sorted out the Higgs scalars which can be utilized in the Higgs mechanism and in the Yukawa sector. We studied the charges and the decompositions of these Higgs scalars. We constructed a suitable Higgs Lagrangian whose potential part consisted of those Higgs multiplets which we regarded as physically $most\ relevant$. These Higgs multiplets were summarized in Table (11.1). We have illustrated in Fig. (11.1), how these multiplets relate certain descents in the Higgs mechanism. We studied the minimum of the potential part of the Higgs Lagrangian, and showed that the minimum of the Higgs field describes a left-right asymmetric vacuum. Since the Higgs couplings were unknown, we couldn't evaluate the vacuum expectation values from the minimum of the Higgs potential despite of the fact that we have been able to solve the vevs from the minimum. On the other side, we studied the kinetic part of the Higgs Lagrangian and obtained a mass-squared matrix of the gauge fields. From the mass-squared matrix, we derived the mass-eigenstates and mass-eigenvalues of the physical gauge fields. We have shown that these expressions, which contain the mixing parameters and phases due to a complex mixing, depend on the vevs and the coupling strengths given by the $SO(10)$ theory. After that we found the physical gauge fields, we have been able to reexpress the physical currents of the $SO(10)$ theory by replacing the gauge fields with the physical gauge fields in the relevant interaction Lagrangian. Consequently, we obtained the charged currents (CC) in the $SO(10)$ theory which are responsible for weak interactions: In this extended version of the weak interaction Lagrangian, the physical gauge fields $W^{\pm}_{1,2}$ coupled not only to left- but also to right-handed currents, but with different strengths. We have shown that the lighter couples dominantly to left-handed currents and the heavier dominantly to right-handed currents. We have shown that this weak interaction Lagrangian is not C, P and CP invariant.

We have found that there are two neutral currents $(NC's)$ in the $SO(10)$ theory. These $NC's$ are mediated by the $Z_{1,2}$ physical gauge fields. We obtained the expressions of the vector- and axial-vector couplings of these neutral currents in terms of the vevs and coupling strengths. We have shown that the vector- and axial-vector couplings of the lighter gauge field are in very good agreement with those predicted by the Standard Model. We concluded that the neutral current mediated by the heavier Z_2 is a new (NC) current beyond the Standard Model. We obtained $SO(10)$ currents that simultaneously carry charge and color, which are responsible for nucleon decays. We briefly considered how these currents can account for the observed baryon asymmetry in nature through investigating the partial decay rates of scalars into quarks and leptons.

Beside the Higgs-sector, we studied the Yukawa-sector. We derived explicit expressions for fermion masses of the third generation by coupling the Higgs fields to the fermions through a democratic Yukawa coupling. We obtained expressions for the Dirac masses of the charged fermions. On the other side, we obtained expressions for the Dirac and Majorana masses of neutrinos. We introduced a flavor-eigenbasis for neutrinos and derived the mass-eigenstates and mass-eigenvalues of the neutrinos from a neutrino mass matrix. We have shown that the physical neutrinos are not purely left- or right-handed but are rather mixtures, however we found that the mixing is extremely small. We found explicit expressions for CP violation in the neutrino sector.

A remarkable fact about $SO(10)$ grand unification is that the parameter space is under determined. This means the number of observables are more than the number of input parameters which determine the values of these observables. The main shortcoming of $SO(10)$ grand unification is our ignorance about the Higgs couplings. This compels us to determine the vevs and their phases indirectly. Another shortcoming of the $SO(10)$ gauge theory is the determination of relative coupling strengths like g_L, g_R and g_{B-L} at mass scales, at which the renormalization equations are no more exact. One can overcome this problem by extrapolating the values. To acquire the values of the vevs and the value of the grand unification coupling strength and as well as the relative coupling strengths we used two sources:

First, we determined the vacuum expectation values and the coupling strengths of gauge interactions given by the $SO(10)$ theory through studying the mass scales in $SO(10)$, in the framework of coupling unification. Complementarily, we determined the vacuum expectation values and their phases by adjusting them to the masses of the known gauge bosons and fermions below the Fermi scale which are accurately measured and known.

In the remaining part of our conclusion, we find it appropriate to briefly review the numerical values of various important observables that we mainly derived in the previous chapter. Through our Ansatz : $v_L \; v_R = G^2$ with $v_R = 10^{10.5\pm0.5}$ GeV from Table 14.1, we have been able to predict the quark and lepton masses at the grand unification mass scale $M_G = 3.38 \cdot 10^{16}$ GeV via a democratic Yukawa coupling. We have renormalized these fermion masses using the QCD renormalization group equations and found that the $SO(10)$ theory successfully predicts the masses of the third generation as summarized in Table 15.3. Using the same input values in Table 15.4, we obtained the masses of all the physical gauge bosons. We have been able to reproduce the W and the Z masses of the Standard Model in very good agreement with the experimental results, where the former was identified as the W_1 and the latter as the Z_1 physical gauge boson. The masses of the heavier W_2 and the Z_2 are found to assume the values $4.03 \cdot 10^9$ GeV and $8.32 \cdot 10^9$ GeV respectively.

We have found the mixing angle between the W_L and the W_R gauge fields as $\xi_1 \cong 0.02631$, which tells us that the CC currents predicted by the $SO(10)$ theory are not in conflict with the current experimental status. We have found that the color and charge carrying gauge fields, which we denoted with $W_3, \ldots, W_7$ have the mass $1.76 \cdot 10^{16}$ GeV, except for W_6 which is lighter and assumes the mass $2.74 \cdot 10^{15}$ GeV. Since W_6 mediates $B - L$ violating interactions, we conclude that it gives rise to relatively faster nucleon decay processes in the leptonic channel. These processes will have a decay rate $\sim 10^4$ times faster.

We have obtained the CP-phases that enter the interaction Lagrangians using the same input values in Table 15.4. We have found that the CP-violating phase in the CC currents and as well as in the colored and charged currents equals to $\pi/4$.

We find it appropriate to mention here that the values of the angles θ, δ and α, which we determined as in in Table 15.4, have almost no influence on the gauge boson masses and the mixing parameters. In contrast they influence the fermion masses very strongly. Therefore a careful study of the angles θ, δ and α might improve the value of the CP-violating phase accordingly. This should be investigated further.

The fermion masses of the first and second generation can not be found unless we relinquish to use the democratic Yukawa coupling. We are convinced that the masses of the first and second generation can be obtained through breaking the flavor symmetry by introducing small perturbation terms into the democratic Yukawa coupling as suggested by H. Fritzsch and D. Holtmannspotter in ref. [77]. The Breaking of the subnuclear democracy might be the origin of flavor mixing. This should be considered further in the framework of the $SO(10)$ gauge theory.

APPENDIX

A. CONVENTIONS AND TOOLS

Quantum numbers of Tensors

A tensor $\Psi^{kl\ldots}_{p\ldots}$ transforming as

$$\Psi^{kl\ldots}_{pq\ldots} \rightarrow U^k_m\, U^l_n\, U^r_p\, U^s_q\, \ldots \Psi^{mn\ldots}_{rs\ldots} \tag{A.1}$$

where $U^k_m = [\, \exp\, (\, i\, \Sigma_{ab}\, \omega_{ab})\,]^k_m$ has the same quantum numbers of a tensor with $\Psi^k \Psi^l \Psi_p$. The charges of Ψ^i are

$$Q\ (\Psi_i) = \ Q_i\, \delta_{ij} \qquad \text{and} \qquad Q\ \left(\Psi^i\right) = -Q_i\, \delta_{ij} \tag{A.2}$$

Herein δ_{ij} is the delta-Kroenecker and Q is the electric charge eigenvalue operator and Q_i its i^{th} eigenvalue. The quantum numbers of $\Psi^k \Psi^l \Psi_p$ are

$$Q\ \left(\Psi^k \Psi^l \Psi_p\right) = -Q_k - Q_l + Q_p \tag{A.3}$$

We can also have instead Q, one of the $B - L$, L_3, R_3 and Y eigenvalue operators in $SO(10)$. The quantum numbers of a 32×32 matrix Ω^i_j at each matrix site of the 45 representation are given in Appendix C.

Tensor Product of Matrices

If A and B are $n \times n$ and $m \times m$ matrices then the tensor product

$$C = A \times B \tag{A.4}$$

is an $mn \times mn$ matrix with elements

$$C_{\alpha\beta} = A_{kl} B_{ij}\ , \qquad \alpha = m(k-1) + i, \qquad \beta = m(l-1) + j \tag{A.5}$$

or if A and B are 2 by 2 matrices

$$A \times B = \begin{pmatrix} a_{11}B & a_{12}B \\ a_{21}B & a_{22}B \end{pmatrix} = \begin{pmatrix} a_{11}\, b_{11} & a_{11}\, b_{12} & a_{12}\, b_{11} & a_{12}\, b_{12} \\ a_{11}\, b_{21} & a_{11}\, b_{22} & a_{12}\, b_{21} & a_{12}\, b_{22} \\ a_{21}\, b_{11} & a_{21}\, b_{12} & a_{22}\, b_{11} & a_{22}\, b_{12} \\ a_{21}\, b_{21} & a_{21}\, b_{22} & a_{22}\, b_{21} & a_{22}\, b_{22} \end{pmatrix} \tag{A.6}$$

The product is not commutative but associative.

Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad , \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad , \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{A.7}$$

Dirac Representation

$$\gamma_0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad , \quad \gamma = \begin{pmatrix} 0 & \sigma \\ \sigma & 0 \end{pmatrix} \quad , \quad \gamma_5 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{A.8}$$

$$(\gamma^0)^\dagger = \gamma^0, \quad (\gamma_5)^\dagger = \gamma_5, \quad (\gamma^k)^\dagger = -\gamma^k \qquad k = 1, 2, 3 \tag{A.9}$$

B. DECOMPOSITIONS OF THE 126

The following complex scalar fields Φ and lowering matrices $\mathbf{\Gamma}$ make the $\Phi \cdot \mathbf{\Gamma} \equiv 126$. In all expressions $\gamma_{ijklm} = \Gamma_i \Gamma_j \Gamma_k \Gamma_l \Gamma_m$ and all ϕ_{ijklm} are real scalar fields where $i, j, k, l, m = (1, \ldots, 10)$ and $\Gamma_m \in SO(10)$ basis. The fields and lowering generators carry the same upper-script as assigned before. Lower-scripts indicating quantum numbers are this time suppressed.

$$\frac{1}{32} Tr\left[(\phi_{ijklm}\Gamma_i\Gamma_j\Gamma_k\Gamma_l\Gamma_m)^2\right] = \sum_{i,j,k,l,m=1}^{10} \phi^2_{ijklm} \quad , \qquad i \neq j \neq k \neq l \neq m \tag{B.1}$$

We start with $\downarrow\uparrow$ states of $(2, 2, 15)$ (see eq. 7.10).

$$\begin{aligned}
\Phi^1 &= +i\,\phi_{12359} + \phi_{12369} + \phi_{12459} - i\,\phi_{12469} + i\,\phi_{35789} + \phi_{36789} + \phi_{45789} - i\,\phi_{46789} \\
&\quad -\phi_{123510} + i\,\phi_{123610} + i\,\phi_{124510} + \phi_{124610} - \phi_{357810} + i\,\phi_{367810} + i\,\phi_{457810} + \phi_{467810} \\
\Gamma^1 &= +i\,\gamma_{12359} + \gamma_{12369} + \gamma_{12459} - i\,\gamma_{12469} + i\,\gamma_{35789} + \gamma_{36789} + \gamma_{45789} - i\,\gamma_{46789} \\
&\quad -\gamma_{123510} + i\,\gamma_{123610} + i\,\gamma_{124510} + \gamma_{124610} - \gamma_{357810} + i\,\gamma_{367810} + i\,\gamma_{457810} + \gamma_{467810} \\
\Phi^2 &= -i\,\phi_{13459} - \phi_{13469} - i\,\phi_{15789} - \phi_{16789} - \phi_{23459} + i\,\phi_{23469} - \phi_{25789} + i\,\phi_{26789} \\
&\quad +\phi_{134510} - i\,\phi_{134610} + \phi_{157810} - i\,\phi_{167810} - i\,\phi_{234510} - \phi_{234610} - i\,\phi_{257810} - \phi_{267810} \\
\Gamma^2 &= -i\,\gamma_{13459} - \gamma_{13469} - i\,\gamma_{15789} - \gamma_{16789} - \gamma_{23459} + i\,\gamma_{23469} - \gamma_{25789} + i\,\gamma_{26789} \\
&\quad +\gamma_{134510} - i\,\gamma_{134610} + \gamma_{157810} - i\,\gamma_{167810} - i\,\gamma_{234510} - \gamma_{234610} - i\,\gamma_{257810} - \gamma_{267810} \\
\Phi^3 &= -i\,\phi_{13569} + i\,\phi_{13789} + \phi_{14569} - \phi_{14789} - \phi_{23569} + \phi_{23789} - i\,\phi_{24569} + i\,\phi_{24789} \\
&\quad +\phi_{135610} - \phi_{137810} + i\,\phi_{145610} - i\,\phi_{147810} - i\,\phi_{235610} + i\,\phi_{237810} + \phi_{245610} - \phi_{247810} \\
\Gamma^3 &= -i\,\gamma_{13569} + i\,\gamma_{13789} + \gamma_{14569} - \gamma_{14789} - \gamma_{23569} + \gamma_{23789} - i\,\gamma_{24569} + i\,\gamma_{24789} \\
&\quad +\gamma_{135610} - \gamma_{137810} + i\,\gamma_{145610} - i\,\gamma_{147810} - i\,\gamma_{235610} + i\,\gamma_{237810} + \gamma_{245610} - \gamma_{247810} \\
\Phi^4 &= -i\,\phi_{12359} + \phi_{12369} + \phi_{12459} + i\,\phi_{12469} - i\,\phi_{35789} + \phi_{36789} + \phi_{45789} + i\,\phi_{46789} \\
&\quad +\phi_{123510} + i\,\phi_{123610} + i\,\phi_{124510} - \phi_{124610} + \phi_{357810} + i\,\phi_{367810} + i\,\phi_{457810} - \phi_{467810} \\
\Gamma^4 &= -i\,\gamma_{12359} + \gamma_{12369} + \gamma_{12459} + i\,\gamma_{12469} - i\,\gamma_{35789} + \gamma_{36789} + \gamma_{45789} + i\,\gamma_{46789} \\
&\quad +\gamma_{123510} + i\,\gamma_{123610} + i\,\gamma_{124510} - \gamma_{124610} + \gamma_{357810} + i\,\gamma_{367810} + i\,\gamma_{457810} - \gamma_{467810} \\
\Phi^5 &= +i\,\phi_{13459} - \phi_{13469} + i\,\phi_{15789} - \phi_{16789} - \phi_{23459} - i\,\phi_{23469} - \phi_{25789} - i\,\phi_{26789} \\
&\quad -\phi_{134510} - i\,\phi_{134610} - \phi_{157810} - i\,\phi_{167810} - i\,\phi_{234510} + \phi_{234610} - i\,\phi_{257810} + \phi_{267810} \\
\Gamma^5 &= +i\,\gamma_{13459} - \gamma_{13469} + i\,\gamma_{15789} - \gamma_{16789} - \gamma_{23459} - i\,\gamma_{23469} - \gamma_{25789} - i\,\gamma_{26789} \\
&\quad -\gamma_{134510} - i\,\gamma_{134610} - \gamma_{157810} - i\,\gamma_{167810} - i\,\gamma_{234510} + \gamma_{234610} - i\,\gamma_{257810} + \gamma_{267810} \\
\Phi^6 &= +i\,\phi_{13569} - i\,\phi_{13789} + \phi_{14569} - \phi_{14789} - \phi_{23569} + \phi_{23789} + i\,\phi_{24569} - i\,\phi_{24789} \\
&\quad -\phi_{135610} + \phi_{137810} + i\,\phi_{145610} - i\,\phi_{147810} - +i\,\phi_{235610} + i\,\phi_{237810} - \phi_{245610} + \phi_{247810} \\
\Gamma^6 &= +i\,\gamma_{13569} - i\,\gamma_{13789} + \gamma_{14569} - \gamma_{14789} - \gamma_{23569} + \gamma_{23789} + i\,\gamma_{24569} - i\,\gamma_{24789} \\
&\quad -\gamma_{135610} + \gamma_{137810} + i\,\gamma_{145610} - i\,\gamma_{147810} - +i\,\gamma_{235610} + i\,\gamma_{237810} - \gamma_{245610} + \gamma_{247810} \\
\Phi^7 &= -\phi_{12349} - \phi_{12789} - \phi_{34569} - \phi_{56789} - i\,\phi_{123410} - i\,\phi_{127810} - i\,\phi_{345610} - i\,\phi_{567810} \\
\Gamma^7 &= -\gamma_{12349} - \gamma_{12789} - \gamma_{34569} - \gamma_{56789} - i\,\gamma_{123410} - i\,\gamma_{127810} - i\,\gamma_{345610} - i\,\gamma_{567810} \\
\Phi^8 &= +\phi_{12569} - \phi_{12789} - \phi_{34569} + \phi_{34789} + i\,\phi_{125610} - i\,\phi_{127810} - i\,\phi_{345610} + i\,\phi_{347810} \\
\Gamma^8 &= +\gamma_{12569} - \gamma_{12789} - \gamma_{34569} + \gamma_{34789} + i\,\gamma_{125610} - i\,\gamma_{127810} - i\,\gamma_{345610} + i\,\gamma_{347810} \\
\Phi^9 &= +i\,\phi_{13569} + i\,\phi_{13789} + \phi_{14569} + \phi_{14789} + \phi_{23569} + \phi_{23789} - i\,\phi_{24569} - i\,\phi_{24789} \\
&\quad -\phi_{135610} - \phi_{137810} + i\,\phi_{145610} + i\,\phi_{147810} + i\,\phi_{235610} + i\,\phi_{237810} + \phi_{245610} + \phi_{247810} \\
\Gamma^9 &= +i\,\gamma_{13569} + i\,\gamma_{13789} + \gamma_{14569} + \gamma_{14789} + \gamma_{23569} + \gamma_{23789} - i\,\gamma_{24569} - i\,\gamma_{24789} \\
&\quad -\gamma_{135610} - \gamma_{137810} + i\,\gamma_{145610} + i\,\gamma_{147810} + i\,\gamma_{235610} + i\,\gamma_{237810} + \gamma_{245610} + \gamma_{247810} \\
\Phi^{10} &= -i\,\phi_{13459} + \phi_{13469} + i\,\phi_{15789} - \phi_{16789} - \phi_{23459} - i\,\phi_{23469} + \phi_{25789} + i\,\phi_{26789} \\
&\quad +\phi_{134510} + i\,\phi_{134610} - \phi_{157810} - i\,\phi_{167810} - i\,\phi_{234510} + \phi_{234610} + i\,\phi_{257810} - \phi_{267810} \\
\Gamma^{10} &= -i\,\gamma_{13459} + \gamma_{13469} + i\,\gamma_{15789} - \gamma_{16789} - \gamma_{23459} - i\,\gamma_{23469} + \gamma_{25789} + i\,\gamma_{26789} \\
&\quad +\gamma_{134510} + i\,\gamma_{134610} - \gamma_{157810} - i\,\gamma_{167810} - i\,\gamma_{234510} + \gamma_{234610} + i\,\gamma_{257810} - \gamma_{267810} \\
\Phi^{11} &= -i\,\phi_{12359} + \phi_{12369} - \phi_{12459} - i\,\phi_{12469} + i\,\phi_{35789} - \phi_{36789} + \phi_{45789} + i\,\phi_{46789} \\
&\quad +\phi_{123510} + i\,\phi_{123610} - i\,\phi_{124510} + \phi_{124610} - \phi_{357810} - i\,\phi_{367810} + i\,\phi_{457810} - \phi_{467810} \\
\Gamma^{11} &= -i\,\gamma_{12359} + \gamma_{12369} - \gamma_{12459} - i\,\gamma_{12469} + i\,\gamma_{35789} - \gamma_{36789} + \gamma_{45789} + i\,\gamma_{46789} \\
&\quad +\gamma_{123510} + i\,\gamma_{123610} - i\,\gamma_{124510} + \gamma_{124610} - \gamma_{357810} - i\,\gamma_{367810} + i\,\gamma_{457810} - \gamma_{467810} \\
\Phi^{12} &= -i\,\phi_{13569} - i\,\phi_{13789} + \phi_{14569} + \phi_{14789} + \phi_{23569} + \phi_{23789} + i\,\phi_{24569} + i\,\phi_{24789} \\
&\quad +\phi_{135610} + \phi_{137810} + i\,\phi_{145610} + i\,\phi_{147810} + i\,\phi_{235610} + i\,\phi_{237810} - \phi_{245610} - \phi_{247810} \\
\Gamma^{12} &= -i\,\gamma_{13569} - i\,\gamma_{13789} + \gamma_{14569} + \gamma_{14789} + \gamma_{23569} + \gamma_{23789} + i\,\gamma_{24569} + i\,\gamma_{24789} \\
&\quad +\gamma_{135610} + \gamma_{137810} + i\,\gamma_{145610} + i\,\gamma_{147810} + i\,\gamma_{235610} + i\,\gamma_{237810} - \gamma_{245610} - \gamma_{247810} \\
\Phi^{13} &= +i\,\phi_{13459} + \phi_{13469} - i\,\phi_{15789} - \phi_{16789} - \phi_{23459} + i\,\phi_{23469} + \phi_{25789} - i\,\phi_{26789} \\
&\quad -\phi_{134510} + i\,\phi_{134610} + \phi_{157810} - i\,\phi_{167810} - i\,\phi_{234510} - \phi_{234610} + i\,\phi_{257810} + \phi_{267810} \\
\Gamma^{13} &= +i\,\gamma_{13459} + \gamma_{13469} - i\,\gamma_{15789} - \gamma_{16789} - \gamma_{23459} + i\,\gamma_{23469} + \gamma_{25789} - i\,\gamma_{26789} \\
&\quad -\gamma_{134510} + i\,\gamma_{134610} + \gamma_{157810} - i\,\gamma_{167810} - i\,\gamma_{234510} - \gamma_{234610} + i\,\gamma_{257810} + \gamma_{267810} \\
\Phi^{14} &= +i\,\phi_{12359} + \phi_{12369} - \phi_{12459} + i\,\phi_{12469} - i\,\phi_{35789} - \phi_{36789} + \phi_{45789} - i\,\phi_{46789} \\
&\quad -\phi_{123510} + i\,\phi_{123610} - i\,\phi_{124510} - \phi_{124610} + \phi_{357810} - i\,\phi_{367810} + i\,\phi_{457810} + \phi_{467810} \\
\Gamma^{14} &= +i\,\gamma_{12359} + \gamma_{12369} - \gamma_{12459} + i\,\gamma_{12469} - i\,\gamma_{35789} - \gamma_{36789} + \gamma_{45789} - i\,\gamma_{46789} \\
&\quad -\gamma_{123510} + i\,\gamma_{123610} - i\,\gamma_{124510} - \gamma_{124610} + \gamma_{357810} - i\,\gamma_{367810} + i\,\gamma_{457810} + \gamma_{467810} \\
\Phi^{15} &= -\phi_{12349} + \phi_{12569} + \phi_{12789} + \phi_{34569} + \phi_{34789} - \phi_{56789} - i\,\phi_{123410} + i\,\phi_{125610} \\
&\quad +i\,\phi_{127810} + i\,\phi_{345610} + i\,\phi_{347810} - i\,\phi_{567810} \\
\Gamma^{15} &= -\gamma_{12349} + \gamma_{12569} + \gamma_{12789} + \gamma_{34569} + \gamma_{34789} - \gamma_{56789} - i\,\gamma_{123410} + i\,\gamma_{125610} \\
&\quad +i\,\gamma_{127810} + i\,\gamma_{345610} + i\,\gamma_{347810} - i\,\gamma_{567810}
\end{aligned}$$

The following complex scalar fields and lowering matrices make the $(\downarrow\downarrow)$ states $(2, 2, 15)$ (see eq. 7.10)

$$\begin{aligned}
\Phi^1 &= +i\,\phi_{12357} - \phi_{12358} + \phi_{12367} + i\,\phi_{12368} + \phi_{12457} + i\,\phi_{12458} - i\,\phi_{12467} + \phi_{12468} \\
&\quad +i\,\phi_{357910} - \phi_{358910} + \phi_{367910} + i\,\phi_{368910} + \phi_{457910} + i\,\phi_{458910} - i\,\phi_{467910} + \phi_{468910} \\
\Gamma^1 &= +i\,\gamma_{12357} - \gamma_{12358} + \gamma_{12367} + i\,\gamma_{12368} + \gamma_{12457} + i\,\gamma_{12458} - i\,\gamma_{12467} + \gamma_{12468} \\
&\quad +i\,\gamma_{357910} - \gamma_{358910} + \gamma_{367910} + i\,\gamma_{368910} + \gamma_{457910} + i\,\gamma_{458910} - i\,\gamma_{467910} + \gamma_{468910} \\
\Phi^2 &= -i\,\phi_{13457} + \phi_{13458} - \phi_{13467} - i\,\phi_{13468} - \phi_{23457} - i\,\phi_{23458} + i\,\phi_{23467} - \phi_{23468} \\
&\quad -i\,\phi_{157910} + \phi_{158910} - \phi_{167910} - i\,\phi_{168910} - \phi_{257910} - i\,\phi_{258910} + i\,\phi_{267910} - \phi_{268910} \\
\Gamma^2 &= -i\,\gamma_{13457} + \gamma_{13458} - \gamma_{13467} - i\,\gamma_{13468} - \gamma_{23457} - i\,\gamma_{23458} + i\,\gamma_{23467} - \gamma_{23468} \\
&\quad -i\,\gamma_{157910} + \gamma_{158910} - \gamma_{167910} - i\,\gamma_{168910} - \gamma_{257910} - i\,\gamma_{258910} + i\,\gamma_{267910} - \gamma_{268910} \\
\Phi^3 &= -i\,\phi_{13567} + \phi_{13568} + \phi_{14567} + i\,\phi_{14568} - \phi_{23567} - i\,\phi_{23568} - i\,\phi_{24567} + \phi_{24568} \\
&\quad +i\,\phi_{137910} - \phi_{138910} - \phi_{147910} - i\,\phi_{148910} + \phi_{237910} + i\,\phi_{238910} + i\,\phi_{247910} - \phi_{248910} \\
\Gamma^3 &= -i\,\gamma_{13567} + \gamma_{13568} + \gamma_{14567} + i\,\gamma_{14568} - \gamma_{23567} - i\,\gamma_{23568} - i\,\gamma_{24567} + \gamma_{24568} \\
&\quad +i\,\gamma_{137910} - \gamma_{138910} - \gamma_{147910} - i\,\gamma_{148910} + \gamma_{237910} + i\,\gamma_{238910} + i\,\gamma_{247910} - \gamma_{248910}
\end{aligned}$$

$$
\begin{array}{lcl}
\Phi^4 & = & -i\,\phi_{12357} + \phi_{12358} + \phi_{12367} + i\,\phi_{12368} + \phi_{12457} + i\,\phi_{12458} + i\,\phi_{12467} - \phi_{12468} \\
 & & -i\,\phi_{357910} + \phi_{358910} + \phi_{367910} + i\,\phi_{368910} + \phi_{457910} + i\,\phi_{458910} + i\,\phi_{467910} - \phi_{468910} \\
\Gamma^4 & = & -i\,\gamma_{12357} + \gamma_{12358} + \gamma_{12367} + i\,\gamma_{12368} + \gamma_{12457} + i\,\gamma_{12458} + i\,\gamma_{12467} - \gamma_{12468} \\
 & & -i\,\gamma_{357910} + \gamma_{358910} + \gamma_{367910} + i\,\gamma_{368910} + \gamma_{457910} + i\,\gamma_{458910} + i\,\gamma_{467910} - \gamma_{468910} \\
\Phi^5 & = & +i\,\phi_{13457} - \phi_{13458} - \phi_{13467} - i\,\phi_{13468} - \phi_{23457} - i\,\phi_{23458} - i\,\phi_{23467} + \phi_{23468} \\
 & & +i\,\phi_{157910} - \phi_{158910} - \phi_{167910} - i\,\phi_{168910} - \phi_{257910} - i\,\phi_{258910} - i\,\phi_{267910} + \phi_{268910} \\
\Gamma^5 & = & +i\,\gamma_{13457} - \gamma_{13458} - \gamma_{13467} - i\,\gamma_{13468} - \gamma_{23457} - i\,\gamma_{23458} - i\,\gamma_{23467} + \gamma_{23468} \\
 & & +i\,\gamma_{157910} - \gamma_{158910} - \gamma_{167910} - i\,\gamma_{168910} - \gamma_{257910} - i\,\gamma_{258910} - i\,\gamma_{267910} + \gamma_{268910} \\
\Phi^6 & = & +i\,\phi_{13567} - \phi_{13568} + \phi_{14567} + i\,\phi_{14568} - \phi_{23567} - i\,\phi_{23568} + i\,\phi_{24567} - \phi_{24568} \\
 & & -i\,\phi_{137910} + \phi_{138910} - \phi_{147910} - i\,\phi_{148910} + \phi_{237910} + i\,\phi_{238910} - i\,\phi_{247910} + \phi_{248910} \\
\Gamma^6 & = & +i\,\gamma_{13567} - \gamma_{13568} + \gamma_{14567} + i\,\gamma_{14568} - \gamma_{23567} - i\,\gamma_{23568} + i\,\gamma_{24567} - \gamma_{24568} \\
 & & -i\,\gamma_{137910} + \gamma_{138910} - \gamma_{147910} - i\,\gamma_{148910} + \gamma_{237910} + i\,\gamma_{238910} - i\,\gamma_{247910} + \gamma_{248910} \\
\Phi^7 & = & -\phi_{12347} - i\,\phi_{12348} - \phi_{34567} - i\,\phi_{34568} - \phi_{127910} - i\,\phi_{128910} - \phi_{567910} - i\,\phi_{568910} \\
\Gamma^7 & = & -\gamma_{12347} - i\,\gamma_{12348} - \gamma_{34567} - i\,\gamma_{34568} - \gamma_{127910} - i\,\gamma_{128910} - \gamma_{567910} - i\,\gamma_{568910} \\
\Phi^8 & = & +\phi_{12567} + i\,\phi_{12568} - \phi_{34567} - i\,\phi_{34568} - \phi_{127910} - i\,\phi_{128910} + \phi_{347910} + i\,\phi_{348910} \\
\gamma^8 & = & +\gamma_{12567} + i\,\gamma_{12568} - \gamma_{34567} - i\,\gamma_{34568} - \gamma_{127910} - i\,\gamma_{128910} + \gamma_{347910} + i\,\gamma_{348910} \\
\Phi^9 & = & +i\,\phi_{13567} - \phi_{13568} + \phi_{14567} + i\,\phi_{14568} + \phi_{23567} + i\,\phi_{23568} - i\,\phi_{24567} + \phi_{24568} \\
 & & +i\,\phi_{137910} - \phi_{138910} + \phi_{147910} + i\,\phi_{148910} + \phi_{237910} + i\,\phi_{238910} - i\,\phi_{247910} + \phi_{248910} \\
\Gamma^9 & = & +i\,\gamma_{13567} - \gamma_{13568} + \gamma_{14567} + i\,\gamma_{14568} + \gamma_{23567} + i\,\gamma_{23568} - i\,\gamma_{24567} + \gamma_{24568} \\
 & & +i\,\gamma_{137910} - \gamma_{138910} + \gamma_{147910} + i\,\gamma_{148910} + \gamma_{237910} + i\,\gamma_{238910} - i\,\gamma_{247910} + \gamma_{248910} \\
\Phi^{10} & = & -i\,\phi_{13457} + \phi_{13458} + \phi_{13467} + i\,\phi_{13468} - \phi_{23457} - i\,\phi_{23458} - i\,\phi_{23467} + \phi_{23468} \\
 & & +i\,\phi_{157910} - \phi_{158910} - \phi_{167910} - i\,\phi_{168910} + \phi_{257910} + i\,\phi_{258910} + i\,\phi_{267910} - \phi_{268910} \\
\Gamma^{10} & = & -i\,\gamma_{13457} + \gamma_{13458} + \gamma_{13467} + i\,\gamma_{13468} - \gamma_{23457} - i\,\gamma_{23458} - i\,\gamma_{23467} + \gamma_{23468} \\
 & & +i\,\gamma_{157910} - \gamma_{158910} - \gamma_{167910} - i\,\gamma_{168910} + \gamma_{257910} + i\,\gamma_{258910} + i\,\gamma_{267910} - \gamma_{268910} \\
\Phi^{11} & = & -i\,\phi_{12357} + \phi_{12358} + \phi_{12367} + i\,\phi_{12368} - \phi_{12457} - i\,\phi_{12458} - i\,\phi_{12467} + \phi_{12468} \\
 & & +i\,\phi_{357910} - \phi_{358910} - \phi_{367910} - i\,\phi_{368910} + \phi_{457910} + i\,\phi_{458910} + i\,\phi_{467910} - \phi_{468910} \\
\Gamma^{11} & = & -i\,\gamma_{12357} + \gamma_{12358} + \gamma_{12367} + i\,\gamma_{12368} - \gamma_{12457} - i\,\gamma_{12458} - i\,\gamma_{12467} + \gamma_{12468} \\
 & & +i\,\gamma_{357910} - \gamma_{358910} - \gamma_{367910} - i\,\gamma_{368910} + \gamma_{457910} + i\,\gamma_{458910} + i\,\gamma_{467910} - \gamma_{468910} \\
\Phi^{12} & = & -i\,\phi_{13567} + \phi_{13568} + \phi_{14567} + i\,\phi_{14568} + \phi_{23567} + i\,\phi_{23568} + i\,\phi_{24567} - \phi_{24568} \\
 & & -i\,\phi_{137910} + \phi_{138910} + \phi_{147910} + i\,\phi_{148910} + \phi_{237910} + i\,\phi_{238910} + i\,\phi_{247910} - \phi_{248910} \\
\Gamma^{12} & = & -i\,\gamma_{13567} + \gamma_{13568} + \gamma_{14567} + i\,\gamma_{14568} + \gamma_{23567} + i\,\gamma_{23568} + i\,\gamma_{24567} - \gamma_{24568} \\
 & & -i\,\gamma_{137910} + \gamma_{138910} + \gamma_{147910} + i\,\gamma_{148910} + \gamma_{237910} + i\,\gamma_{238910} + i\,\gamma_{247910} - \gamma_{248910} \\
\Phi^{13} & = & +i\,\phi_{13457} - \phi_{13458} + \phi_{13467} + i\,\phi_{13468} - \phi_{23457} - i\,\phi_{23458} + i\,\phi_{23467} - \phi_{23468} \\
 & & -i\,\phi_{157910} + \phi_{158910} - \phi_{167910} - i\,\phi_{168910} + \phi_{257910} + i\,\phi_{258910} - i\,\phi_{267910} + \phi_{268910} \\
\Gamma^{13} & = & +i\,\gamma_{13457} - \gamma_{13458} + \gamma_{13467} + i\,\gamma_{13468} - \gamma_{23457} - i\,\gamma_{23458} + i\,\gamma_{23467} - \gamma_{23468} \\
 & & -i\,\gamma_{157910} + \gamma_{158910} - \gamma_{167910} - i\,\gamma_{168910} + \gamma_{257910} + i\,\gamma_{258910} - i\,\gamma_{267910} + \gamma_{268910} \\
\Phi^{14} & = & +i\,\phi_{12357} - \phi_{12358} + \phi_{12367} + i\,\phi_{12368} - \phi_{12457} - i\,\phi_{12458} + i\,\phi_{12467} - \phi_{12468} \\
 & & -i\,\phi_{357910} + \phi_{358910} - \phi_{367910} - i\,\phi_{368910} + \phi_{457910} + i\,\phi_{458910} - i\,\phi_{467910} + \phi_{468910} \\
\Gamma^{14} & = & +i\,\gamma_{12357} - \gamma_{12358} + \gamma_{12367} + i\,\gamma_{12368} - \gamma_{12457} - i\,\gamma_{12458} + i\,\gamma_{12467} - \gamma_{12468} \\
 & & -i\,\gamma_{357910} + \gamma_{358910} - \gamma_{367910} - i\,\gamma_{368910} + \gamma_{457910} + i\,\gamma_{458910} - i\,\gamma_{467910} + \gamma_{468910} \\
\Phi^{15} & = & -\phi_{12347} - i\,\phi_{12348} + \phi_{12567} + i\,\phi_{12568} + \phi_{34567} + i\,\phi_{34568} + \phi_{127910} + i\,\phi_{128910} \\
 & & +\phi_{347910} + i\,\phi_{348910} - \phi_{567910} - i\,\phi_{568910} \\
\gamma^{15} & = & -\gamma_{12347} - i\,\gamma_{12348} + \gamma_{12567} + i\,\gamma_{12568} + \gamma_{34567} + i\,\gamma_{34568} + \gamma_{127910} + i\,\gamma_{128910} \\
 & & +\gamma_{347910} + i\,\gamma_{348910} - \gamma_{567910} - i\,\gamma_{568910}
\end{array}
$$

The following complex scalar fields and lowering matrices make the $(\uparrow\downarrow)$ states $(2,2,15)$ (see eq. 7.10)

$$
\begin{array}{lcl}
\Phi^1 & = & +i\,\gamma_{12359} + \phi_{12369} + \phi_{12459} - i\,\phi_{12469} + i\,\phi_{35789} + \phi_{36789} + \phi_{45789} - i\,\phi_{46789} \\
 & & -\phi_{123510} + i\,\phi_{123610} + i\,\phi_{124510} + \phi_{124610} - \phi_{357810} + i\,\phi_{367810} + i\,\phi_{457810} + \phi_{467810} \\
\Gamma^1 & = & +i\,\gamma_{12359} + \gamma_{12369} + \gamma_{12459} - i\,\gamma_{12469} + i\,\gamma_{35789} + \gamma_{36789} + \gamma_{45789} - i\,\gamma_{46789} \\
 & & -\gamma_{123510} + i\,\gamma_{123610} + i\,\gamma_{124510} + \gamma_{124610} - \gamma_{357810} + i\,\gamma_{367810} + i\,\gamma_{457810} + \gamma_{467810} \\
\Phi^2 & = & -i\,\phi_{13459} - \phi_{13469} - i\,\phi_{15789} - \phi_{16789} - \phi_{23459} + i\,\phi_{23469} - \phi_{25789} + i\,\phi_{26789} \\
 & & +\phi_{134510} - i\,\phi_{134610} + \phi_{157810} - i\,\phi_{167810} - i\,\phi_{234510} - \phi_{234610} - i\,\phi_{257810} - \phi_{267810} \\
\Gamma^2 & = & -i\,\gamma_{13459} - \gamma_{13469} - i\,\gamma_{15789} - \gamma_{16789} - \gamma_{23459} + i\,\gamma_{23469} - \gamma_{25789} + i\,\gamma_{26789} \\
 & & +\gamma_{134510} - i\,\gamma_{134610} + \gamma_{157810} - i\,\gamma_{167810} - i\,\gamma_{234510} - \gamma_{234610} - i\,\gamma_{257810} - \gamma_{267810} \\
\Phi^3 & = & -i\,\phi_{13569} + i\,\phi_{13789} + \phi_{14569} - \phi_{14789} - \phi_{23569} + \phi_{23789} - i\,\phi_{24569} + i\,\phi_{24789} \\
 & & +\phi_{135610} - \phi_{137810} + i\,\phi_{145610} - i\,\phi_{147810} - +i\,\phi_{235610} + i\,\phi_{237810} + \phi_{245610} - \phi_{247810} \\
\Gamma^3 & = & -i\,\gamma_{13569} + i\,\gamma_{13789} + \gamma_{14569} - \gamma_{14789} - \gamma_{23569} + \gamma_{23789} - i\,\gamma_{24569} + i\,\gamma_{24789} \\
 & & +\gamma_{135610} - \gamma_{137810} + i\,\gamma_{145610} - i\,\gamma_{147810} - +i\,\gamma_{235610} + i\,\gamma_{237810} + \gamma_{245610} - \gamma_{247810} \\
\Phi^4 & = & -i\,\phi_{12359} + \phi_{12369} + \phi_{12459} + i\,\phi_{12469} - i\,\phi_{35789} + \phi_{36789} + \phi_{45789} + i\,\phi_{46789} \\
 & & +\phi_{123510} + i\,\phi_{123610} + i\,\phi_{124510} - \phi_{124610} + \phi_{357810} + i\,\phi_{367810} + i\,\phi_{457810} - \phi_{467810} \\
\Gamma^4 & = & -i\,\gamma_{12359} + \gamma_{12369} + \gamma_{12459} + i\,\gamma_{12469} - i\,\gamma_{35789} + \gamma_{36789} + \gamma_{45789} + i\,\gamma_{46789} \\
 & & +\gamma_{123510} + i\,\gamma_{123610} + i\,\gamma_{124510} - \gamma_{124610} + \gamma_{357810} + i\,\gamma_{367810} + i\,\gamma_{457810} - \gamma_{467810} \\
\Phi^5 & = & +i\,\phi_{13459} - \phi_{13469} + i\,\phi_{15789} - \phi_{16789} - \phi_{23459} - i\,\phi_{23469} - \phi_{25789} - i\,\phi_{26789} \\
 & & -\phi_{134510} - i\,\phi_{134610} - \phi_{157810} - i\,\phi_{167810} - i\,\phi_{234510} + \phi_{234610} - i\,\phi_{257810} + \phi_{267810} \\
\Gamma^5 & = & +i\,\gamma_{13459} - \gamma_{13469} + i\,\gamma_{15789} - \gamma_{16789} - \gamma_{23459} - i\,\gamma_{23469} - \gamma_{25789} - i\,\gamma_{26789} \\
 & & -\gamma_{134510} - i\,\gamma_{134610} - \gamma_{157810} - i\,\gamma_{167810} - i\,\gamma_{234510} + \gamma_{234610} - i\,\gamma_{257810} + \gamma_{267810} \\
\Phi^6 & = & +i\,\phi_{13569} - i\,\phi_{13789} + \phi_{14569} - \phi_{14789} - \phi_{23569} + \phi_{23789} + i\,\phi_{24569} - i\,\phi_{24789} \\
 & & -\phi_{135610} + \phi_{137810} + i\,\phi_{145610} - i\,\phi_{147810} - i\,\phi_{235610} + i\,\phi_{237810} - \phi_{245610} + \phi_{247810} \\
\Gamma^6 & = & +i\,\gamma_{13569} - i\,\gamma_{13789} + \gamma_{14569} - \gamma_{14789} - \gamma_{23569} + \gamma_{23789} + i\,\gamma_{24569} - i\,\gamma_{24789} \\
 & & -\gamma_{135610} + \gamma_{137810} + i\,\gamma_{145610} - i\,\gamma_{147810} - i\,\gamma_{235610} + i\,\gamma_{237810} - \gamma_{245610} + \gamma_{247810} \\
\Phi^7 & = & -\phi_{12349} - \phi_{12789} - \phi_{34569} - \phi_{56789} - i\,\phi_{123410} - i\,\phi_{127810} - +i\,\phi_{345610} - i\,\phi_{567810} \\
\Gamma^7 & = & -\gamma_{12349} - \gamma_{12789} - \gamma_{34569} - \gamma_{56789} - i\,\gamma_{123410} - i\,\gamma_{127810} - +i\,\gamma_{345610} - i\,\gamma_{567810} \\
\Phi^8 & = & +\phi_{12569} - \phi_{12789} - \phi_{34569} + \phi_{34789} + i\,\phi_{125610} - i\,\phi_{127810} - i\,\phi_{345610} + i\,\phi_{347810} \\
\Gamma^8 & = & +\gamma_{12569} - \gamma_{12789} - \gamma_{34569} + \gamma_{34789} + i\,\gamma_{125610} - i\,\gamma_{127810} - i\,\gamma_{345610} + i\,\gamma_{347810} \\
\Phi^9 & = & +i\,\phi_{13569} + i\,\phi_{13789} + \phi_{14569} + \phi_{14789} + \phi_{23569} + \phi_{23789} - i\,\phi_{24569} - i\,\phi_{24789} \\
 & & -\phi_{135610} - \phi_{137810} + i\,\phi_{145610} + i\,\phi_{147810} + i\,\phi_{235610} + i\,\phi_{237810} + \phi_{245610} + \phi_{247810} \\
\Gamma^9 & = & +i\,\gamma_{13569} + i\,\gamma_{13789} + \gamma_{14569} + \gamma_{14789} + \gamma_{23569} + \gamma_{23789} - i\,\gamma_{24569} - i\,\gamma_{24789} \\
 & & -\gamma_{135610} - \gamma_{137810} + i\,\gamma_{145610} + i\,\gamma_{147810} + i\,\gamma_{235610} + i\,\gamma_{237810} + \gamma_{245610} + \gamma_{247810} \\
\Phi^{10} & = & -i\,\phi_{13459} + \phi_{13469} + i\,\phi_{15789} - \phi_{16789} - \phi_{23459} - i\,\phi_{23469} + \phi_{25789} + i\,\phi_{26789} \\
 & & +\phi_{134510} + i\,\phi_{134610} - \phi_{157810} - i\,\phi_{167810} - i\,\phi_{234510} + \phi_{234610} + i\,\phi_{257810} - \phi_{267810} \\
\Gamma^{10} & = & -i\,\gamma_{13459} + \gamma_{13469} + i\,\gamma_{15789} - \gamma_{16789} - \gamma_{23459} - i\,\gamma_{23469} + \gamma_{25789} + i\,\gamma_{26789} \\
 & & +\gamma_{134510} + i\,\gamma_{134610} - \gamma_{157810} - i\,\gamma_{167810} - i\,\gamma_{234510} + \gamma_{234610} + i\,\gamma_{257810} - \gamma_{267810} \\
\Phi^{11} & = & -i\,\phi_{12359} + \phi_{12369} - \phi_{12459} - i\,\phi_{12469} + i\,\phi_{35789} - \phi_{36789} + \phi_{45789} + i\,\phi_{46789} \\
 & & +\phi_{123510} + i\,\phi_{123610} - i\,\phi_{124510} + \phi_{124610} - \phi_{357810} - i\,\phi_{367810} + i\,\phi_{457810} - \phi_{467810} \\
\Gamma^{11} & = & -i\,\gamma_{12359} + \gamma_{12369} - \gamma_{12459} - i\,\gamma_{12469} + i\,\gamma_{35789} - \gamma_{36789} + \gamma_{45789} + i\,\gamma_{46789} \\
 & & +\gamma_{123510} + i\,\gamma_{123610} - i\,\gamma_{124510} + \gamma_{124610} - \gamma_{357810} - i\,\gamma_{367810} + i\,\gamma_{457810} - \gamma_{467810} \\
\Phi^{12} & = & -i\,\phi_{13569} - i\,\phi_{13789} + \phi_{14569} + \phi_{14789} + \phi_{23569} + \phi_{23789} + i\,\phi_{24569} + i\,\phi_{24789} \\
 & & +\phi_{135610} + \phi_{137810} + i\,\phi_{145610} + i\,\phi_{147810} + i\,\phi_{235610} + i\,\phi_{237810} - \phi_{245610} - \phi_{247810} \\
\Gamma^{12} & = & -i\,\gamma_{13569} - i\,\gamma_{13789} + \gamma_{14569} + \gamma_{14789} + \gamma_{23569} + \gamma_{23789} + i\,\gamma_{24569} + i\,\gamma_{24789} \\
 & & +\gamma_{135610} + \gamma_{137810} + i\,\gamma_{145610} + i\,\gamma_{147810} + i\,\gamma_{235610} + i\,\gamma_{237810} - \gamma_{245610} - \gamma_{247810} \\
\Phi^{13} & = & +i\,\phi_{13459} + \phi_{13469} - i\,\phi_{15789} - \phi_{16789} - \phi_{23459} + i\,\phi_{23469} + \phi_{25789} - i\,\phi_{26789} \\
 & & -\phi_{134510} + i\,\phi_{134610} + \phi_{157810} - i\,\phi_{167810} - i\,\phi_{234510} - \phi_{234610} + i\,\phi_{257810} + \phi_{267810} \\
\Gamma^{13} & = & +i\,\gamma_{13459} + \gamma_{13469} - i\,\gamma_{15789} - \gamma_{16789} - \gamma_{23459} + i\,\gamma_{23469} + \gamma_{25789} - i\,\gamma_{26789} \\
 & & -\gamma_{134510} + i\,\gamma_{134610} + \gamma_{157810} - i\,\gamma_{167810} - i\,\gamma_{234510} - \gamma_{234610} + i\,\gamma_{257810} + \gamma_{267810} \\
\Phi^{14} & = & +i\,\phi_{12359} + \phi_{12369} - \phi_{12459} + i\,\phi_{12469} - i\,\phi_{35789} - \phi_{36789} + \phi_{45789} - i\,\phi_{46789} \\
 & & -\phi_{123510} + i\,\phi_{123610} - i\,\phi_{124510} - \phi_{124610} + \phi_{357810} - i\,\phi_{367810} + i\,\phi_{457810} + \phi_{467810} \\
\Gamma^{14} & = & +i\,\gamma_{12359} + \gamma_{12369} - \gamma_{12459} + i\,\gamma_{12469} - i\,\gamma_{35789} - \gamma_{36789} + \gamma_{45789} - i\,\gamma_{46789} \\
 & & -\gamma_{123510} + i\,\gamma_{123610} - i\,\gamma_{124510} - \gamma_{124610} + \gamma_{357810} - i\,\gamma_{367810} + i\,\gamma_{457810} + \gamma_{467810} \\
\Phi^{15} & = & -\phi_{12349} + \phi_{12569} + \phi_{12789} + \phi_{34569} + \phi_{34789} - \phi_{56789} - i\,\phi_{123410} + i\,\phi_{125610} \\
 & & +i\,\phi_{127810} + i\,\phi_{345610} + i\,\phi_{347810} - i\,\phi_{567810} \\
\Gamma^{15} & = & -\gamma_{12349} + \gamma_{12569} + \gamma_{12789} + \gamma_{34569} + \gamma_{34789} - \gamma_{56789} - i\,\gamma_{123410} + i\,\gamma_{125610} \\
 & & +i\,\gamma_{127810} + i\,\gamma_{345610} + i\,\gamma_{347810} - i\,\gamma_{567810}
\end{array}
$$

The following complex scalar fields and lowering matrices make the $(\uparrow\uparrow)$ states $(2,2,15)$ (see eq. 7.10)

$$
\begin{aligned}
\Phi^1 &= +i\,\phi_{12357} + \phi_{12358} + \phi_{12367} - i\,\phi_{12368} + \phi_{12457} - i\,\phi_{12458} - i\,\phi_{12467} - \phi_{12468} \\
&\quad -i\,\phi_{357910} - \phi_{358910} - \phi_{367910} + i\,\phi_{368910} - \phi_{457910} + i\,\phi_{458910} + i\,\phi_{467910} + \phi_{468910} \\
\Gamma^1 &= +i\,\gamma_{12357} + \gamma_{12358} + \gamma_{12367} - i\,\gamma_{12368} + \gamma_{12457} - i\,\gamma_{12458} - i\,\gamma_{12467} - \gamma_{12468} \\
&\quad -i\,\gamma_{357910} - \gamma_{358910} - \gamma_{367910} + i\,\gamma_{368910} - \gamma_{457910} + i\,\gamma_{458910} + i\,\gamma_{467910} + \gamma_{468910} \\
\Phi^2 &= -i\,\phi_{13457} - \phi_{13458} - \phi_{13467} + i\,\phi_{13468} - \phi_{23457} + i\,\phi_{23458} + i\,\phi_{23467} + \phi_{23468} \\
&\quad +i\,\phi_{157910} + \phi_{158910} + \phi_{167910} - i\,\phi_{168910} + \phi_{257910} - i\,\phi_{258910} - i\,\phi_{267910} - \phi_{268910} \\
\Gamma^2 &= -i\,\gamma_{13457} - \gamma_{13458} - \gamma_{13467} + i\,\gamma_{13468} - \gamma_{23457} + i\,\gamma_{23458} + i\,\gamma_{23467} + \gamma_{23468} \\
&\quad +i\,\gamma_{157910} + \gamma_{158910} + \gamma_{167910} - i\,\gamma_{168910} + \gamma_{257910} - i\,\gamma_{258910} - i\,\gamma_{267910} - \gamma_{268910} \\
\Phi^3 &= -i\,\phi_{13567} - \phi_{13568} + \phi_{14567} - i\,\phi_{14568} - \phi_{23567} + i\,\phi_{23568} - i\,\phi_{24567} - \phi_{24568} \\
&\quad -i\,\phi_{137910} - \phi_{138910} + \phi_{147910} - i\,\phi_{148910} - \phi_{237910} + i\,\phi_{238910} - i\,\phi_{247910} - \phi_{248910} \\
\Gamma^3 &= -i\,\gamma_{13567} - \gamma_{13568} + \gamma_{14567} - i\,\gamma_{14568} - \gamma_{23567} + i\,\gamma_{23568} - i\,\gamma_{24567} - \gamma_{24568} \\
&\quad -i\,\gamma_{137910} - \gamma_{138910} + \gamma_{147910} - i\,\gamma_{148910} - \gamma_{237910} + i\,\gamma_{238910} - i\,\gamma_{247910} - \gamma_{248910} \\
\Phi^4 &= -i\,\phi_{12357} - \phi_{12358} + \phi_{12367} - i\,\phi_{12368} + \phi_{12457} - i\,\phi_{12458} + i\,\phi_{12467} + \phi_{12468} \\
&\quad +i\,\phi_{357910} + \phi_{358910} - \phi_{367910} + i\,\phi_{368910} - \phi_{457910} + i\,\phi_{458910} - i\,\phi_{467910} - \phi_{468910} \\
\Gamma^4 &= -i\,\gamma_{12357} - \gamma_{12358} + \gamma_{12367} - i\,\gamma_{12368} + \gamma_{12457} - i\,\gamma_{12458} + i\,\gamma_{12467} + \gamma_{12468} \\
&\quad +i\,\gamma_{357910} + \gamma_{358910} - \gamma_{367910} + i\,\gamma_{368910} - \gamma_{457910} + i\,\gamma_{458910} - i\,\gamma_{467910} - \gamma_{468910} \\
\Phi^5 &= +i\,\phi_{13457} + \phi_{13458} - \phi_{13467} + i\,\phi_{13468} - \phi_{23457} + i\,\phi_{23458} - i\,\phi_{23467} - \phi_{23468} \\
&\quad -i\,\phi_{157910} - \phi_{158910} + \phi_{167910} - i\,\phi_{168910} + \phi_{257910} - i\,\phi_{258910} + i\,\phi_{267910} + \phi_{268910} \\
\Gamma^5 &= +i\,\gamma_{13457} + \gamma_{13458} - \gamma_{13467} + i\,\gamma_{13468} - \gamma_{23457} + i\,\gamma_{23458} - i\,\gamma_{23467} - \gamma_{23468} \\
&\quad -i\,\gamma_{157910} - \gamma_{158910} + \gamma_{167910} - i\,\gamma_{168910} + \gamma_{257910} - i\,\gamma_{258910} + i\,\gamma_{267910} + \gamma_{268910} \\
\Phi^6 &= +i\,\phi_{13567} + \phi_{13568} + \phi_{14567} - i\,\phi_{14568} - \phi_{23567} + i\,\phi_{23568} + i\,\phi_{24567} + \phi_{24568} \\
&\quad +i\,\phi_{137910} + \phi_{138910} + \phi_{147910} - i\,\phi_{148910} - \phi_{237910} + i\,\phi_{238910} + i\,\phi_{247910} + \phi_{248910} \\
\Gamma^6 &= +i\,\gamma_{13567} + \gamma_{13568} + \gamma_{14567} - i\,\gamma_{14568} - \gamma_{23567} + i\,\gamma_{23568} + i\,\gamma_{24567} + \gamma_{24568} \\
&\quad +i\,\gamma_{137910} + \gamma_{138910} + \gamma_{147910} - i\,\gamma_{148910} - \gamma_{237910} + i\,\gamma_{238910} + i\,\gamma_{247910} + \gamma_{248910} \\
\Phi^7 &= -\phi_{12347} + i\,\phi_{12348} - \phi_{34567} + i\,\phi_{34568} + \phi_{127910} - i\,\phi_{128910} + \phi_{567910} - i\,\phi_{568910} \\
\Gamma^7 &= -\gamma_{12347} + i\,\gamma_{12348} - \gamma_{34567} + i\,\gamma_{34568} + \gamma_{127910} - i\,\gamma_{128910} + \gamma_{567910} - i\,\gamma_{568910} \\
\Phi^8 &= +\phi_{12567} - i\,\phi_{12568} - \phi_{34567} + i\,\phi_{34568} + \phi_{127910} - i\,\phi_{128910} - \phi_{347910} + i\,\phi_{348910} \\
\Gamma^8 &= +\gamma_{12567} - i\,\gamma_{12568} - \gamma_{34567} + i\,\gamma_{34568} + \gamma_{127910} - i\,\gamma_{128910} - \gamma_{347910} + i\,\gamma_{348910} \\
\Phi^9 &= +i\,\phi_{13567} + \phi_{13568} + \phi_{14567} - i\,\phi_{14568} + \phi_{23567} - i\,\phi_{23568} - i\,\phi_{24567} - \phi_{24568} \\
&\quad -i\,\phi_{137910} - \phi_{138910} - \phi_{147910} + i\,\phi_{148910} - \phi_{237910} + i\,\phi_{238910} + i\,\phi_{247910} + \phi_{248910} \\
\Gamma^9 &= +i\,\gamma_{13567} + \gamma_{13568} + \gamma_{14567} - i\,\gamma_{14568} + \gamma_{23567} - i\,\gamma_{23568} - i\,\gamma_{24567} - \gamma_{24568} \\
&\quad -i\,\gamma_{137910} - \gamma_{138910} - \gamma_{147910} + i\,\gamma_{148910} - \gamma_{237910} + i\,\gamma_{238910} + i\,\gamma_{247910} + \gamma_{248910} \\
\Phi^{10} &= -i\,\phi_{13457} - \phi_{13458} + \phi_{13467} - i\,\phi_{13468} - \phi_{23457} + i\,\phi_{23458} - i\,\phi_{23467} - \phi_{23468} \\
&\quad -i\,\phi_{157910} - \phi_{158910} + \phi_{167910} - i\,\phi_{168910} - \phi_{257910} + i\,\phi_{258910} - i\,\phi_{267910} - \phi_{268910} \\
\Gamma^{10} &= -i\,\gamma_{13457} - \gamma_{13458} + \gamma_{13467} - i\,\gamma_{13468} - \gamma_{23457} + i\,\gamma_{23458} - i\,\gamma_{23467} - \gamma_{23468} \\
&\quad -i\,\gamma_{157910} - \gamma_{158910} + \gamma_{167910} - i\,\gamma_{168910} - \gamma_{257910} + i\,\gamma_{258910} - i\,\gamma_{267910} - \gamma_{268910} \\
\Phi^{11} &= -i\,\phi_{12357} - \phi_{12358} + \phi_{12367} - i\,\phi_{12368} - \phi_{12457} + i\,\phi_{12458} - i\,\phi_{12467} - \phi_{12468} \\
&\quad -i\,\phi_{357910} - \phi_{358910} + \phi_{367910} - i\,\phi_{368910} - \phi_{457910} + i\,\phi_{458910} - i\,\phi_{467910} - \phi_{468910} \\
\Gamma^{11} &= -i\,\gamma_{12357} - \gamma_{12358} + \gamma_{12367} - i\,\gamma_{12368} - \gamma_{12457} + i\,\gamma_{12458} - i\,\gamma_{12467} - \gamma_{12468} \\
&\quad -i\,\gamma_{357910} - \gamma_{358910} + \gamma_{367910} - i\,\gamma_{368910} - \gamma_{457910} + i\,\gamma_{458910} - i\,\gamma_{467910} - \gamma_{468910} \\
\Phi^{12} &= -i\,\phi_{13567} - \phi_{13568} + \phi_{14567} - i\,\phi_{14568} + \phi_{23567} - i\,\phi_{23568} + i\,\phi_{24567} + \phi_{24568} \\
&\quad +i\,\phi_{137910} + \phi_{138910} - \phi_{147910} + i\,\phi_{148910} - \phi_{237910} + i\,\phi_{238910} - i\,\phi_{247910} - \phi_{248910} \\
\Gamma^{12} &= -i\,\gamma_{13567} - \gamma_{13568} + \gamma_{14567} - i\,\gamma_{14568} + \gamma_{23567} - i\,\gamma_{23568} + i\,\gamma_{24567} + \gamma_{24568} \\
&\quad +i\,\gamma_{137910} + \gamma_{138910} - \gamma_{147910} + i\,\gamma_{148910} - \gamma_{237910} + i\,\gamma_{238910} - i\,\gamma_{247910} - \gamma_{248910} \\
\Phi^{13} &= +i\,\phi_{13457} + \phi_{13458} + \phi_{13467} - i\,\phi_{13468} - \phi_{23457} + i\,\phi_{23458} + i\,\phi_{23467} + \phi_{23468} \\
&\quad +i\,\phi_{157910} + \phi_{158910} + \phi_{167910} - i\,\phi_{168910} - \phi_{257910} + i\,\phi_{258910} + i\,\phi_{267910} + \phi_{268910} \\
\Gamma^{13} &= +i\,\gamma_{13457} + \gamma_{13458} + \gamma_{13467} - i\,\gamma_{13468} - \gamma_{23457} + i\,\gamma_{23458} + i\,\gamma_{23467} + \gamma_{23468} \\
&\quad +i\,\gamma_{157910} + \gamma_{158910} + \gamma_{167910} - i\,\gamma_{168910} - \gamma_{257910} + i\,\gamma_{258910} + i\,\gamma_{267910} + \gamma_{268910} \\
\Phi^{14} &= +i\,\phi_{12357} + \phi_{12358} + \phi_{12367} - i\,\phi_{12368} - \phi_{12457} + i\,\phi_{12458} + i\,\phi_{12467} + \phi_{12468} \\
&\quad +i\,\phi_{357910} + \phi_{358910} + \phi_{367910} - i\,\phi_{368910} - \phi_{457910} + i\,\phi_{458910} + i\,\phi_{467910} + \phi_{468910} \\
\Gamma^{14} &= +i\,\gamma_{12357} + \gamma_{12358} + \gamma_{12367} - i\,\gamma_{12368} - \gamma_{12457} + i\,\gamma_{12458} + i\,\gamma_{12467} + \gamma_{12468} \\
&\quad +i\,\gamma_{357910} + \gamma_{358910} + \gamma_{367910} - i\,\gamma_{368910} - \gamma_{457910} + i\,\gamma_{458910} + i\,\gamma_{467910} + \gamma_{468910} \\
\Phi^{15} &= -\phi_{12347} + i\,\phi_{12348} + \phi_{12567} - i\,\phi_{12568} + \phi_{34567} - i\,\phi_{34568} - \phi_{127910} + i\,\phi_{128910} \\
&\quad -\phi_{347910} + i\,\phi_{348910} + \phi_{567910} - i\,\phi_{568910} \\
\Gamma^{15} &= -\gamma_{12347} + i\,\gamma_{12348} + \gamma_{12567} - i\,\gamma_{12568} + \gamma_{34567} - i\,\gamma_{34568} - \gamma_{127910} + i\,\gamma_{128910} \\
&\quad -\gamma_{347910} + i\,\gamma_{348910} + \gamma_{567910} - i\,\gamma_{568910}
\end{aligned}
$$

The following complex scalar fields and lowering matrices make the $(0\uparrow)$ states $(1,3,10)$ (see eq. 7.3)

$$
\begin{aligned}
\Phi^1 &= -\phi_{13479} + i\,\phi_{13489} - \phi_{15679} + i\,\phi_{15689} + i\,\phi_{23479} + \phi_{23489} + i\,\phi_{25679} + \phi_{25689} \\
&\quad -i\,\phi_{134710} - \phi_{134810} - i\,\phi_{156710} - \phi_{156810} - \phi_{234710} + i\,\phi_{234810} - \phi_{256710} + i\,\phi_{256810} \\
\Gamma^1 &= -\gamma_{13479} + i\,\gamma_{13489} - \gamma_{15679} + i\,\gamma_{15689} + i\,\gamma_{23479} + \gamma_{23489} + i\,\gamma_{25679} + \gamma_{25689} \\
&\quad -i\,\gamma_{134710} - \gamma_{134810} - i\,\gamma_{156710} - \gamma_{156810} - \gamma_{234710} + i\,\gamma_{234810} - \gamma_{256710} + i\,\gamma_{256810} \\
\Phi^2 &= -\phi_{12379} + i\,\phi_{12389} + i\,\phi_{12479} + \phi_{12489} - \phi_{35679} + i\,\phi_{35689} + i\,\phi_{45679} + \phi_{45689} \\
&\quad -i\,\phi_{123710} - \phi_{123810} - \phi_{124710} + i\,\phi_{124810} - i\,\phi_{356710} - \phi_{356810} - \phi_{456710} + i\,\phi_{456810} \\
\Gamma^2 &= -\gamma_{12379} + i\,\gamma_{12389} + i\,\gamma_{12479} + \gamma_{12489} - \gamma_{35679} + i\,\gamma_{35689} + i\,\gamma_{45679} + \gamma_{45689} \\
&\quad -i\,\gamma_{123710} - \gamma_{123810} - \gamma_{124710} + i\,\gamma_{124810} - i\,\gamma_{356710} - \gamma_{356810} - \gamma_{456710} + i\,\gamma_{456810} \\
\Phi^3 &= -\phi_{12579} + i\,\phi_{12589} + i\,\phi_{12679} + \phi_{12689} - \phi_{34579} + i\,\phi_{34589} + i\,\phi_{34679} + \phi_{34689} \\
&\quad -i\,\phi_{125710} - \phi_{125810} - \phi_{126710} + i\,\phi_{126810} - i\,\phi_{345710} - \phi_{345810} - \phi_{346710} + i\,\phi_{346810} \\
\Gamma^3 &= -\gamma_{12579} + i\,\gamma_{12589} + i\,\gamma_{12679} + \gamma_{12689} - \gamma_{34579} + i\,\gamma_{34589} + i\,\gamma_{34679} + \gamma_{34689} \\
&\quad -i\,\gamma_{125710} - \gamma_{125810} - \gamma_{126710} + i\,\gamma_{126810} - i\,\gamma_{345710} - \gamma_{345810} - \gamma_{346710} + i\,\gamma_{346810} \\
\Phi^4 &= -\phi_{13479} + i\,\phi_{13489} + \phi_{15679} - i\,\phi_{15689} - i\,\phi_{23479} - \phi_{23489} + i\,\phi_{25679} + \phi_{25689} \\
&\quad -i\,\phi_{134710} - \phi_{134810} + i\,\phi_{156710} + \phi_{156810} + \phi_{234710} - i\,\phi_{234810} - \phi_{256710} + i\,\phi_{256810} \\
\Gamma^4 &= -\gamma_{13479} + i\,\gamma_{13489} + \gamma_{15679} - i\,\gamma_{15689} - i\,\gamma_{23479} - \gamma_{23489} + i\,\gamma_{25679} + \gamma_{25689} \\
&\quad -i\,\gamma_{134710} - \gamma_{134810} + i\,\gamma_{156710} + \gamma_{156810} + \gamma_{234710} - i\,\gamma_{234810} - \gamma_{256710} + i\,\gamma_{256810} \\
\Phi^5 &= +\phi_{12379} - i\,\phi_{12389} + i\,\phi_{12479} + \phi_{12489} - \phi_{35679} + i\,\phi_{35689} - i\,\phi_{45679} - \phi_{45689} \\
&\quad +i\,\phi_{123710} + \phi_{123810} - \phi_{124710} + i\,\phi_{124810} - i\,\phi_{356710} - \phi_{356810} + \phi_{456710} - i\,\phi_{456810} \\
\Gamma^5 &= +\gamma_{12379} - i\,\gamma_{12389} + i\,\gamma_{12479} + \gamma_{12489} - \gamma_{35679} + i\,\gamma_{35689} - i\,\gamma_{45679} - \gamma_{45689} \\
&\quad +i\,\gamma_{123710} + \gamma_{123810} - \gamma_{124710} + i\,\gamma_{124810} - i\,\gamma_{356710} - \gamma_{356810} + \gamma_{456710} - i\,\gamma_{456810} \\
\Phi^6 &= -\phi_{12579} + i\,\phi_{12589} - i\,\phi_{12679} - \phi_{12689} + \phi_{34579} - i\,\phi_{34589} + i\,\phi_{34679} + \phi_{34689} \\
&\quad -i\,\phi_{125710} - \phi_{125810} + \phi_{126710} - i\,\phi_{126810} + i\,\phi_{345710} + \phi_{345810} - \phi_{346710} + i\,\phi_{346810} \\
\Gamma^6 &= -\gamma_{12579} + i\,\gamma_{12589} - i\,\gamma_{12679} - \gamma_{12689} + \gamma_{34579} - i\,\gamma_{34589} + i\,\gamma_{34679} + \gamma_{34689} \\
&\quad -i\,\gamma_{125710} - \gamma_{125810} + \gamma_{126710} - i\,\gamma_{126810} + i\,\gamma_{345710} + \gamma_{345810} - \gamma_{346710} + i\,\gamma_{346810} \\
\Phi^7 &= -i\,\phi_{13579} - \phi_{13589} - \phi_{13679} + i\,\phi_{13689} - \phi_{14579} + i\,\phi_{14589} + i\,\phi_{14679} + \phi_{14689} \\
&\quad -\phi_{23579} + i\,\phi_{23589} + i\,\phi_{23679} + \phi_{23689} + i\,\phi_{24579} + \phi_{24589} + \phi_{24679} - i\,\phi_{24689} \\
&\quad +\phi_{135710} - i\,\phi_{135810} - i\,\phi_{136710} - \phi_{136810} - i\,\phi_{145710} - \phi_{145810} - \phi_{146710} + i\,\phi_{146810} \\
&\quad -i\,\phi_{235710} - \phi_{235810} - \phi_{236710} + i\,\phi_{236810} - \phi_{245710} + i\,\phi_{245810} + i\,\phi_{246710} + \phi_{246810} \\
\Gamma^7 &= -i\,\gamma_{13579} - \gamma_{13589} - \gamma_{13679} + i\,\gamma_{13689} - \gamma_{14579} + i\,\gamma_{14589} + i\,\gamma_{14679} + \gamma_{14689} \\
&\quad -\gamma_{23579} + i\,\gamma_{23589} + i\,\gamma_{23679} + \gamma_{23689} + i\,\gamma_{24579} + \gamma_{24589} + \gamma_{24679} - i\,\gamma_{24689} \\
&\quad +\gamma_{135710} - i\,\gamma_{135810} - i\,\gamma_{136710} - \gamma_{136810} - i\,\gamma_{145710} - \gamma_{145810} - \gamma_{146710} + i\,\gamma_{146810} \\
&\quad -i\,\gamma_{235710} - \gamma_{235810} - \gamma_{236710} + i\,\gamma_{236810} - \gamma_{245710} + i\,\gamma_{245810} + i\,\gamma_{246710} + \gamma_{246810}
\end{aligned}
$$

$$
\begin{aligned}
\Phi^{8} \quad = \quad & -i\,\phi_{13579} - \phi_{13589} + \phi_{13679} - i\,\phi_{13689} + \phi_{14579} - i\,\phi_{14589} + i\,\phi_{14679} + \phi_{14689} \\
& -\phi_{23579} + i\,\phi_{23589} - i\,\phi_{23679} - \phi_{23689} - i\,\phi_{24579} - \phi_{24589} + \phi_{24679} - i\,\phi_{24689} \\
& +\phi_{135710} - i\,\phi_{135810} + i\,\phi_{136710} + \phi_{136810} + i\,\phi_{145710} + \phi_{145810} - \phi_{146710} + i\,\phi_{146810} \\
& -i\,\phi_{235710} - \phi_{235810} + \phi_{236710} - i\,\phi_{236810} + \phi_{245710} - i\,\phi_{245810} + i\,\phi_{246710} + \phi_{246810} \\
\Gamma^{8} \quad = \quad & -i\,\gamma_{13579} - \gamma_{13589} + \gamma_{13679} - i\,\gamma_{13689} + \gamma_{14579} - i\,\gamma_{14589} + i\,\gamma_{14679} + \gamma_{14689} \\
& -\gamma_{23579} + i\,\gamma_{23589} - i\,\gamma_{23679} - \gamma_{23689} - i\,\gamma_{24579} - \gamma_{24589} + \gamma_{24679} - i\,\gamma_{24689} \\
& +\gamma_{135710} - i\,\gamma_{135810} + i\,\gamma_{136710} + \gamma_{136810} + i\,\gamma_{145710} + \gamma_{145810} - \gamma_{146710} + i\,\gamma_{146810} \\
& -i\,\gamma_{235710} - \gamma_{235810} + \gamma_{236710} - i\,\gamma_{236810} + \gamma_{245710} - i\,\gamma_{245810} + i\,\gamma_{246710} + \gamma_{246810} \\
\Phi^{9} \quad = \quad & -i\,\phi_{13579} - \phi_{13589} + \phi_{13679} - i\,\phi_{13689} - \phi_{14579} + i\,\phi_{14589} - i\,\phi_{14679} - \phi_{14689} \\
& +\phi_{23579} - i\,\phi_{23589} + i\,\phi_{23679} + \phi_{23689} - i\,\phi_{24579} - \phi_{24589} + \phi_{24679} - i\,\phi_{24689} \\
& +\phi_{135710} - i\,\phi_{135810} + i\,\phi_{136710} + \phi_{136810} - i\,\phi_{145710} - \phi_{145810} + \phi_{146710} - i\,\phi_{146810} \\
& +i\,\phi_{235710} + \phi_{235810} - \phi_{236710} + i\,\phi_{236810} + \phi_{245710} - i\,\phi_{245810} + i\,\phi_{246710} + \phi_{246810} \\
\Gamma^{9} \quad = \quad & -i\,\gamma_{13579} - \gamma_{13589} + \gamma_{13679} - i\,\gamma_{13689} - \gamma_{14579} + i\,\gamma_{14589} - i\,\gamma_{14679} - \gamma_{14689} \\
& +\gamma_{23579} - i\,\gamma_{23589} + i\,\gamma_{23679} + \gamma_{23689} - i\,\gamma_{24579} - \gamma_{24589} + \gamma_{24679} - i\,\gamma_{24689} \\
& +\gamma_{135710} - i\,\gamma_{135810} + i\,\gamma_{136710} + \gamma_{136810} - i\,\gamma_{145710} - \gamma_{145810} + \gamma_{146710} - i\,\gamma_{146810} \\
& +i\,\gamma_{235710} + \gamma_{235810} - \gamma_{236710} + i\,\gamma_{236810} + \gamma_{245710} - i\,\gamma_{245810} + i\,\gamma_{246710} + \gamma_{246810} \\
\Phi^{10} \quad = \quad & -i\,\phi_{13579} - \phi_{13589} - \phi_{13679} + i\,\phi_{13689} + \phi_{14579} - i\,\phi_{14589} - i\,\phi_{14679} - \phi_{14689} \\
& +\phi_{23579} - i\,\phi_{23589} - i\,\phi_{23679} - \phi_{23689} + i\,\phi_{24579} + \phi_{24589} + \phi_{24679} - i\,\phi_{24689} \\
& +\phi_{135710} - i\,\phi_{135810} - i\,\phi_{136710} - \phi_{136810} + i\,\phi_{145710} + \phi_{145810} + \phi_{146710} - i\,\phi_{146810} \\
& +i\,\phi_{235710} + \phi_{235810} + \phi_{236710} - i\,\phi_{236810} - \phi_{245710} + i\,\phi_{245810} + i\,\phi_{246710} + \phi_{246810} \\
\Gamma^{10} \quad = \quad & -i\,\gamma_{13579} - \gamma_{13589} - \gamma_{13679} + i\,\gamma_{13689} + \gamma_{14579} - i\,\gamma_{14589} - i\,\gamma_{14679} - \gamma_{14689} \\
& +\gamma_{23579} - i\,\gamma_{23589} - i\,\gamma_{23679} - \gamma_{23689} + i\,\gamma_{24579} + \gamma_{24589} + \gamma_{24679} - i\,\gamma_{24689} \\
& +\gamma_{135710} - i\,\gamma_{135810} - i\,\gamma_{136710} - \gamma_{136810} + i\,\gamma_{145710} + \gamma_{145810} + \gamma_{146710} - i\,\gamma_{146810} \\
& +i\,\gamma_{235710} + \gamma_{235810} + \gamma_{236710} - i\,\gamma_{236810} - \gamma_{245710} + i\,\gamma_{245810} + i\,\gamma_{246710} + \gamma_{246810}
\end{aligned}
$$

The following complex scalar fields and lowering matrices make the $(0, \Delta = 0)$ states $(1, 3, 10)$ (see eq. 7.3)

$$
\begin{aligned}
\Phi^{1} \quad = \quad & +i\,\phi_{13478} + i\,\phi_{15678} + \phi_{23478} + \phi_{25678} - i\,\phi_{134910} - i\,\phi_{156910} - \phi_{234910} - \phi_{256910} \\
\Gamma^{1} \quad = \quad & +i\,\gamma_{13478} + i\,\gamma_{15678} + \gamma_{23478} + \gamma_{25678} - i\,\gamma_{134910} - i\,\gamma_{156910} - \gamma_{234910} - \gamma_{256910} \\
\Phi^{2} \quad = \quad & +i\,\phi_{12378} + \phi_{12478} + i\,\phi_{35678} + \phi_{45678} - i\,\phi_{123910} - \phi_{124910} - i\,\phi_{356910} - \phi_{456910} \\
\Gamma^{2} \quad = \quad & +i\,\gamma_{12378} + \gamma_{12478} + i\,\gamma_{35678} + \gamma_{45678} - i\,\gamma_{123910} - \gamma_{124910} - i\,\gamma_{356910} - \gamma_{456910} \\
\Phi^{3} \quad = \quad & +i\,\phi_{12578} + \phi_{12678} + i\,\phi_{34578} + \phi_{34678} - i\,\phi_{125910} - \phi_{126910} - i\,\phi_{345910} - \phi_{346910} \\
\Gamma^{3} \quad = \quad & +i\,\gamma_{12578} + \gamma_{12678} + i\,\gamma_{34578} + \gamma_{34678} - i\,\gamma_{125910} - \gamma_{126910} - i\,\gamma_{345910} - \gamma_{346910} \\
\Phi^{4} \quad = \quad & +i\,\phi_{13478} - i\,\phi_{15678} - \phi_{23478} + \phi_{25678} - i\,\phi_{134910} + i\,\phi_{156910} + \phi_{234910} - \phi_{256910} \\
\Gamma^{4} \quad = \quad & +i\,\gamma_{13478} - i\,\gamma_{15678} - \gamma_{23478} + \gamma_{25678} - i\,\gamma_{134910} + i\,\gamma_{156910} + \gamma_{234910} - \gamma_{256910} \\
\Phi^{5} \quad = \quad & -i\,\phi_{12378} + \phi_{12478} + i\,\phi_{35678} - \phi_{45678} + i\,\phi_{123910} - \phi_{124910} - i\,\phi_{356910} + \phi_{456910} \\
\Gamma^{5} \quad = \quad & -i\,\gamma_{12378} + \gamma_{12478} + i\,\gamma_{35678} - \gamma_{45678} + i\,\gamma_{123910} - \gamma_{124910} - i\,\gamma_{356910} + \gamma_{456910} \\
\Phi^{6} \quad = \quad & +i\,\phi_{12578} - \phi_{12678} - i\,\phi_{34578} + \phi_{34678} - i\,\phi_{125910} + \phi_{126910} + i\,\phi_{345910} - \phi_{346910} \\
\Gamma^{6} \quad = \quad & +i\,\gamma_{12578} - \gamma_{12678} - i\,\gamma_{34578} + \gamma_{34678} - i\,\gamma_{125910} + \gamma_{126910} + i\,\gamma_{345910} - \gamma_{346910} \\
\Phi^{7} \quad = \quad & -\phi_{13578} + i\,\phi_{13678} + i\,\phi_{14578} + \phi_{14678} + i\,\phi_{23578} + \phi_{23678} + \phi_{24578} - i\,\phi_{24678} \\
& +\phi_{135910} - i\,\phi_{136910} - i\,\phi_{145910} - \phi_{146910} - i\,\phi_{235910} - \phi_{236910} - \phi_{245910} + i\,\phi_{246910} \\
\Gamma^{7} \quad = \quad & -\gamma_{13578} + i\,\gamma_{13678} + i\,\gamma_{14578} + \gamma_{14678} + i\,\gamma_{23578} + \gamma_{23678} + \gamma_{24578} - i\,\gamma_{24678} \\
& +\gamma_{135910} - i\,\gamma_{136910} - i\,\gamma_{145910} - \gamma_{146910} - i\,\gamma_{235910} - \gamma_{236910} - \gamma_{245910} + i\,\gamma_{246910} \\
\Phi^{8} \quad = \quad & -\phi_{13578} - i\,\phi_{13678} - i\,\phi_{14578} + \phi_{14678} + i\,\phi_{23578} - \phi_{23678} - \phi_{24578} - i\,\phi_{24678} \\
& +\phi_{135910} + i\,\phi_{136910} + i\,\phi_{145910} - \phi_{146910} - i\,\phi_{235910} + \phi_{236910} + \phi_{245910} + i\,\phi_{246910} \\
\Gamma^{8} \quad = \quad & -\gamma_{13578} - i\,\gamma_{13678} - i\,\gamma_{14578} + \gamma_{14678} + i\,\gamma_{23578} - \gamma_{23678} - \gamma_{24578} - i\,\gamma_{24678} \\
& +\gamma_{135910} + i\,\gamma_{136910} + i\,\gamma_{145910} - \gamma_{146910} - i\,\gamma_{235910} + \gamma_{236910} + \gamma_{245910} + i\,\gamma_{246910} \\
\Phi^{9} \quad = \quad & -\phi_{13578} - i\,\phi_{13678} + i\,\phi_{14578} - \phi_{14678} - i\,\phi_{23578} + \phi_{23678} - \phi_{24578} - i\,\phi_{24678} \\
& +\phi_{135910} + i\,\phi_{136910} - i\,\phi_{145910} + \phi_{146910} + i\,\phi_{235910} - \phi_{236910} + \phi_{245910} + i\,\phi_{246910} \\
\Gamma^{9} \quad = \quad & -\gamma_{13578} - i\,\gamma_{13678} + i\,\gamma_{14578} - \gamma_{14678} - i\,\gamma_{23578} + \gamma_{23678} - \gamma_{24578} - i\,\gamma_{24678} \\
& +\gamma_{135910} + i\,\gamma_{136910} - i\,\gamma_{145910} + \gamma_{146910} + i\,\gamma_{235910} - \gamma_{236910} + \gamma_{245910} + i\,\gamma_{246910} \\
\Phi^{10} \quad = \quad & -\phi_{13578} + i\,\phi_{13678} - i\,\phi_{14578} - \phi_{14678} - i\,\phi_{23578} - \phi_{23678} + \phi_{24578} - i\,\phi_{24678} \\
& +\phi_{135910} - i\,\phi_{136910} + i\,\phi_{145910} + \phi_{146910} + i\,\phi_{235910} + \phi_{236910} - \phi_{245910} + i\,\phi_{246910} \\
\Gamma^{10} \quad = \quad & -\gamma_{13578} + i\,\gamma_{13678} - i\,\gamma_{14578} - \gamma_{14678} - i\,\gamma_{23578} - \gamma_{23678} + \gamma_{24578} - i\,\gamma_{24678} \\
& +\gamma_{135910} - i\,\gamma_{136910} + i\,\gamma_{145910} + \gamma_{146910} + i\,\gamma_{235910} + \gamma_{236910} - \gamma_{245910} + i\,\gamma_{246910}
\end{aligned}
$$

The following complex scalar fields and lowering matrices make the $(0 \downarrow)$ states $(1, 3, 10)$ (see eq. 7.3)

$$
\begin{aligned}
\Phi^{1} \quad = \quad & +\phi_{13479} + i\,\phi_{13489} + \phi_{15679} + i\,\phi_{15689} - i\,\phi_{23479} + \phi_{23489} - i\,\phi_{25679} + \phi_{25689} \\
& -i\,\phi_{134710} + \phi_{134810} - i\,\phi_{156710} + \phi_{156810} - \phi_{234710} - i\,\phi_{234810} - \phi_{256710} - i\,\phi_{256810} \\
\Gamma^{1} \quad = \quad & +\gamma_{13479} + i\,\gamma_{13489} + \gamma_{15679} + i\,\gamma_{15689} - i\,\gamma_{23479} + \gamma_{23489} - i\,\gamma_{25679} + \gamma_{25689} \\
& -i\,\gamma_{134710} + \gamma_{134810} - i\,\gamma_{156710} + \gamma_{156810} - \gamma_{234710} - i\,\gamma_{234810} - \gamma_{256710} - i\,\gamma_{256810} \\
\Phi^{2} \quad = \quad & +\phi_{12379} + i\,\phi_{12389} - i\,\phi_{12479} + \phi_{12489} + \phi_{35679} + i\,\phi_{35689} - i\,\phi_{45679} + \phi_{45689} \\
& -i\,\phi_{123710} + \phi_{123810} - \phi_{124710} - i\,\phi_{124810} - i\,\phi_{356710} + \phi_{356810} - \phi_{456710} - i\,\phi_{456810} \\
\Gamma^{2} \quad = \quad & +\gamma_{12379} + i\,\gamma_{12389} - i\,\gamma_{12479} + \gamma_{12489} + \gamma_{35679} + i\,\gamma_{35689} - i\,\gamma_{45679} + \gamma_{45689} \\
& -i\,\gamma_{123710} + \gamma_{123810} - \gamma_{124710} - i\,\gamma_{124810} - i\,\gamma_{356710} + \gamma_{356810} - \gamma_{456710} - i\,\gamma_{456810} \\
\Phi^{3} \quad = \quad & +\phi_{12579} + i\,\phi_{12589} - i\,\phi_{12679} + \phi_{12689} + \phi_{34579} + i\,\phi_{34589} - i\,\phi_{34679} + \phi_{34689} \\
& -i\,\phi_{125710} + \phi_{125810} - \phi_{126710} - i\,\phi_{126810} - i\,\phi_{345710} + \phi_{345810} - \phi_{346710} - i\,\phi_{346810} \\
\Gamma^{3} \quad = \quad & +\gamma_{12579} + i\,\gamma_{12589} - i\,\gamma_{12679} + \gamma_{12689} + \gamma_{34579} + i\,\gamma_{34589} - i\,\gamma_{34679} + \gamma_{34689} \\
& -i\,\gamma_{125710} + \gamma_{125810} - \gamma_{126710} - i\,\gamma_{126810} - i\,\gamma_{345710} + \gamma_{345810} - \gamma_{346710} - i\,\gamma_{346810} \\
\Phi^{4} \quad = \quad & +\phi_{13479} + i\,\phi_{13489} - \phi_{15679} - i\,\phi_{15689} + i\,\phi_{23479} - \phi_{23489} - i\,\phi_{25679} + \phi_{25689} \\
& -i\,\phi_{134710} + \phi_{134810} + i\,\phi_{156710} - \phi_{156810} + \phi_{234710} + i\,\phi_{234810} - \phi_{256710} - i\,\phi_{256810} \\
\Gamma^{4} \quad = \quad & +\gamma_{13479} + i\,\gamma_{13489} - \gamma_{15679} - i\,\gamma_{15689} + i\,\gamma_{23479} - \gamma_{23489} - i\,\gamma_{25679} + \gamma_{25689} \\
& -i\,\gamma_{134710} + \gamma_{134810} + i\,\gamma_{156710} - \gamma_{156810} + \gamma_{234710} + i\,\gamma_{234810} - \gamma_{256710} - i\,\gamma_{256810} \\
\Phi^{5} \quad = \quad & -\phi_{12379} - i\,\phi_{12389} - i\,\phi_{12479} + \phi_{12489} + \phi_{35679} + i\,\phi_{35689} + i\,\phi_{45679} - \phi_{45689} \\
& +i\,\phi_{123710} - \phi_{123810} - \phi_{124710} - i\,\phi_{124810} - i\,\phi_{356710} + \phi_{356810} + \phi_{456710} + i\,\phi_{456810} \\
\Gamma^{5} \quad = \quad & -\gamma_{12379} - i\,\gamma_{12389} - i\,\gamma_{12479} + \gamma_{12489} + \gamma_{35679} + i\,\gamma_{35689} + i\,\gamma_{45679} - \gamma_{45689} \\
& +i\,\gamma_{123710} - \gamma_{123810} - \gamma_{124710} - i\,\gamma_{124810} - i\,\gamma_{356710} + \gamma_{356810} + \gamma_{456710} + i\,\gamma_{456810} \\
\Phi^{6} \quad = \quad & +\phi_{12579} + i\,\phi_{12589} + i\,\phi_{12679} - \phi_{12689} - \phi_{34579} - i\,\phi_{34589} - i\,\phi_{34679} + \phi_{34689} \\
& -i\,\phi_{125710} + \phi_{125810} + \phi_{126710} + i\,\phi_{126810} + i\,\phi_{345710} - \phi_{345810} - \phi_{346710} - i\,\phi_{346810} \\
\Gamma^{6} \quad = \quad & +\gamma_{12579} + i\,\gamma_{12589} + i\,\gamma_{12679} - \gamma_{12689} - \gamma_{34579} - i\,\gamma_{34589} - i\,\gamma_{34679} + \gamma_{34689} \\
& -i\,\gamma_{125710} + \gamma_{125810} + \gamma_{126710} + i\,\gamma_{126810} + i\,\gamma_{345710} - \gamma_{345810} - \gamma_{346710} - i\,\gamma_{346810} \\
\Phi^{7} \quad = \quad & +i\,\phi_{13579} - \phi_{13589} + \phi_{13679} + i\,\phi_{13689} + \phi_{14579} + i\,\phi_{14589} - i\,\phi_{14679} + \phi_{14689} \\
& +\phi_{23579} + i\,\phi_{23589} - i\,\phi_{23679} + \phi_{23689} - i\,\phi_{24579} + \phi_{24589} - \phi_{24679} - i\,\phi_{24689} \\
& +\phi_{135710} + i\,\phi_{135810} - i\,\phi_{136710} + \phi_{136810} - i\,\phi_{145710} + \phi_{145810} - \phi_{146710} - i\,\phi_{146810} \\
& -i\,\phi_{235710} + \phi_{235810} - \phi_{236710} - i\,\phi_{236810} - \phi_{245710} - i\,\phi_{245810} + i\,\phi_{246710} - \phi_{246810} \\
\Gamma^{7} \quad = \quad & +i\,\gamma_{13579} - \gamma_{13589} + \gamma_{13679} + i\,\gamma_{13689} + \gamma_{14579} + i\,\gamma_{14589} - i\,\gamma_{14679} + \gamma_{14689} \\
& +\gamma_{23579} + i\,\gamma_{23589} - i\,\gamma_{23679} + \gamma_{23689} - i\,\gamma_{24579} + \gamma_{24589} - \gamma_{24679} - i\,\gamma_{24689} \\
& +\gamma_{135710} + i\,\gamma_{135810} - i\,\gamma_{136710} + \gamma_{136810} - i\,\gamma_{145710} + \gamma_{145810} - \gamma_{146710} - i\,\gamma_{146810} \\
& -i\,\gamma_{235710} + \gamma_{235810} - \gamma_{236710} - i\,\gamma_{236810} - \gamma_{245710} - i\,\gamma_{245810} + i\,\gamma_{246710} - \gamma_{246810} \\
\Phi^{8} \quad = \quad & +i\,\phi_{13579} - \phi_{13589} - \phi_{13679} - i\,\phi_{13689} - \phi_{14579} - i\,\phi_{14589} - i\,\phi_{14679} + \phi_{14689} \\
& +\phi_{23579} + i\,\phi_{23589} + i\,\phi_{23679} - \phi_{23689} + i\,\phi_{24579} - \phi_{24589} - \phi_{24679} - i\,\phi_{24689} \\
& +\phi_{135710} + i\,\phi_{135810} + i\,\phi_{136710} - \phi_{136810} + i\,\phi_{145710} - \phi_{145810} - \phi_{146710} - i\,\phi_{146810} \\
& -i\,\phi_{235710} + \phi_{235810} + \phi_{236710} + i\,\phi_{236810} + \phi_{245710} + i\,\phi_{245810} + i\,\phi_{246710} - \phi_{246810} \\
\Gamma^{8} \quad = \quad & +i\,\gamma_{13579} - \gamma_{13589} - \gamma_{13679} - i\,\gamma_{13689} - \gamma_{14579} - i\,\gamma_{14589} - i\,\gamma_{14679} + \gamma_{14689} \\
& +\gamma_{23579} + i\,\gamma_{23589} + i\,\gamma_{23679} - \gamma_{23689} + i\,\gamma_{24579} - \gamma_{24589} - \gamma_{24679} - i\,\gamma_{24689} \\
& +\gamma_{135710} + i\,\gamma_{135810} + i\,\gamma_{136710} - \gamma_{136810} + i\,\gamma_{145710} - \gamma_{145810} - \gamma_{146710} - i\,\gamma_{146810} \\
& -i\,\gamma_{235710} + \gamma_{235810} + \gamma_{236710} + i\,\gamma_{236810} + \gamma_{245710} + i\,\gamma_{245810} + i\,\gamma_{246710} - \gamma_{246810}
\end{aligned}
$$

$$
\begin{aligned}
\Phi^9 &= +i\,\phi_{13579} - \phi_{13589} - \phi_{13679} - i\,\phi_{13689} + \phi_{14579} + i\,\phi_{14589} + i\,\phi_{14679} - \phi_{14689} \\
&\quad -\phi_{23579} - i\,\phi_{23589} - i\,\phi_{23679} + \phi_{23689} + i\,\phi_{24579} - \phi_{24589} - \phi_{24679} - i\,\phi_{24689} \\
&\quad +\phi_{135710} + i\,\phi_{135810} + i\,\phi_{136710} - \phi_{136810} - i\,\phi_{145710} + \phi_{145810} + \phi_{146710} + i\,\phi_{146810} \\
&\quad +i\,\phi_{235710} - \phi_{235810} - \phi_{236710} - i\,\phi_{236810} + \phi_{245710} + i\,\phi_{245810} + i\,\phi_{246710} - \phi_{246810} \\
\Gamma^9 &= +i\,\gamma_{13579} - \gamma_{13589} - \gamma_{13679} - i\,\gamma_{13689} + \gamma_{14579} + i\,\gamma_{14589} + i\,\gamma_{14679} - \gamma_{14689} \\
&\quad -\gamma_{23579} - i\,\gamma_{23589} - i\,\gamma_{23679} + \gamma_{23689} + i\,\gamma_{24579} - \gamma_{24589} - \gamma_{24679} - i\,\gamma_{24689} \\
&\quad +\gamma_{135710} + i\,\gamma_{135810} + i\,\gamma_{136710} - \gamma_{136810} - i\,\gamma_{145710} + \gamma_{145810} + \gamma_{146710} + i\,\gamma_{146810} \\
&\quad +i\,\gamma_{235710} - \gamma_{235810} - \gamma_{236710} - i\,\gamma_{236810} + \gamma_{245710} + i\,\gamma_{245810} + i\,\gamma_{246710} - \gamma_{246810} \\
\Phi^{10} &= +i\,\phi_{13579} - \phi_{13589} + \phi_{13679} + i\,\phi_{13689} - \phi_{14579} - i\,\phi_{14589} + i\,\phi_{14679} - \phi_{14689} \\
&\quad -\phi_{23579} - i\,\phi_{23589} + i\,\phi_{23679} - \phi_{23689} - i\,\phi_{24579} + \phi_{24589} - \phi_{24679} - i\,\phi_{24689} \\
&\quad +\phi_{135710} + i\,\phi_{135810} - i\,\phi_{136710} + \phi_{136810} + i\,\phi_{145710} - \phi_{145810} + \phi_{146710} + i\,\phi_{146810} \\
&\quad +i\,\phi_{235710} - \phi_{235810} + \phi_{236710} + i\,\phi_{236810} - \phi_{245710} - i\,\phi_{245810} + i\,\phi_{246710} - \phi_{246810} \\
\Gamma^{10} &= +i\,\gamma_{13579} - \gamma_{13589} + \gamma_{13679} + i\,\gamma_{13689} - \gamma_{14579} - i\,\gamma_{14589} + i\,\gamma_{14679} - \gamma_{14689} \\
&\quad -\gamma_{23579} - i\,\gamma_{23589} + i\,\gamma_{23679} - \gamma_{23689} - i\,\gamma_{24579} + \gamma_{24589} - \gamma_{24679} - i\,\gamma_{24689} \\
&\quad +\gamma_{135710} + i\,\gamma_{135810} - i\,\gamma_{136710} + \gamma_{136810} + i\,\gamma_{145710} - \gamma_{145810} + \gamma_{146710} + i\,\gamma_{146810} \\
&\quad +i\,\gamma_{235710} - \gamma_{235810} + \gamma_{236710} + i\,\gamma_{236810} - \gamma_{245710} - i\,\gamma_{245810} + i\,\gamma_{246710} - \gamma_{246810}
\end{aligned}
$$

The following complex scalar fields and lowering matrices make the $(\uparrow, 0)$ states $(3, 1, \bar{1}0)$ (see eq. 7.3)

$$
\begin{aligned}
\Phi^1 &= +\phi_{13479} - i\,\phi_{13489} - \phi_{15679} + i\,\phi_{15689} - i\,\phi_{23479} - \phi_{23489} + i\,\phi_{25679} + \phi_{25689} \\
&\quad -i\,\phi_{134710} - \phi_{134810} + i\,\phi_{156710} + \phi_{156810} - \phi_{234710} + i\,\phi_{234810} + \phi_{256710} - i\,\phi_{256810} \\
\Gamma^1 &= +\gamma_{13479} - i\,\gamma_{13489} - \gamma_{15679} + i\,\gamma_{15689} - i\,\gamma_{23479} - \gamma_{23489} + i\,\gamma_{25679} + \gamma_{25689} \\
&\quad -i\,\gamma_{134710} - \gamma_{134810} + i\,\gamma_{156710} + \gamma_{156810} - \gamma_{234710} + i\,\gamma_{234810} + \gamma_{256710} - i\,\gamma_{256810} \\
\Phi^2 &= -\phi_{12379} + i\,\phi_{12389} + i\,\phi_{12479} + \phi_{12489} + \phi_{35679} - i\,\phi_{35689} - i\,\phi_{45679} - \phi_{45689} \\
&\quad +i\,\phi_{123710} + \phi_{123810} + \phi_{124710} - i\,\phi_{124810} - i\,\phi_{356710} - \phi_{356810} - \phi_{456710} + i\,\phi_{456810} \\
\Gamma^2 &= -\gamma_{12379} + i\,\gamma_{12389} + i\,\gamma_{12479} + \gamma_{12489} + \gamma_{35679} - i\,\gamma_{35689} - i\,\gamma_{45679} - \gamma_{45689} \\
&\quad +i\,\gamma_{123710} + \gamma_{123810} + \gamma_{124710} - i\,\gamma_{124810} - i\,\gamma_{356710} - \gamma_{356810} - \gamma_{456710} + i\,\gamma_{456810} \\
\Phi^3 &= +\phi_{12579} - i\,\phi_{12589} - i\,\phi_{12679} - \phi_{12689} - \phi_{34579} + i\,\phi_{34589} + i\,\phi_{34679} + \phi_{34689} \\
&\quad -i\,\phi_{125710} - \phi_{125810} - \phi_{126710} + i\,\phi_{126810} + i\,\phi_{345710} + \phi_{345810} + \phi_{346710} - i\,\phi_{346810} \\
\Gamma^3 &= +\gamma_{12579} - i\,\gamma_{12589} - i\,\gamma_{12679} - \gamma_{12689} - \gamma_{34579} + i\,\gamma_{34589} + i\,\gamma_{34679} + \gamma_{34689} \\
&\quad -i\,\gamma_{125710} - \gamma_{125810} - \gamma_{126710} + i\,\gamma_{126810} + i\,\gamma_{345710} + \gamma_{345810} + \gamma_{346710} - i\,\gamma_{346810} \\
\Phi^4 &= +\phi_{13479} - i\,\phi_{13489} + \phi_{15679} - i\,\phi_{15689} + i\,\phi_{23479} + \phi_{23489} + i\,\phi_{25679} + \phi_{25689} \\
&\quad -i\,\phi_{134710} - \phi_{134810} - i\,\phi_{156710} - \phi_{156810} + \phi_{234710} - i\,\phi_{234810} + \phi_{256710} - i\,\phi_{256810} \\
\Gamma^4 &= +\gamma_{13479} - i\,\gamma_{13489} + \gamma_{15679} - i\,\gamma_{15689} + i\,\gamma_{23479} + \gamma_{23489} + i\,\gamma_{25679} + \gamma_{25689} \\
&\quad -i\,\gamma_{134710} - \gamma_{134810} - i\,\gamma_{156710} - \gamma_{156810} + \gamma_{234710} - i\,\gamma_{234810} + \gamma_{256710} - i\,\gamma_{256810} \\
\Phi^5 &= +\phi_{12379} - i\,\phi_{12389} + i\,\phi_{12479} + \phi_{12489} + \phi_{35679} - i\,\phi_{35689} + i\,\phi_{45679} + \phi_{45689} \\
&\quad -i\,\phi_{123710} - \phi_{123810} + \phi_{124710} - i\,\phi_{124810} - i\,\phi_{356710} - \phi_{356810} + \phi_{456710} - i\,\phi_{456810} \\
\Gamma^5 &= +\gamma_{12379} - i\,\gamma_{12389} + i\,\gamma_{12479} + \gamma_{12489} + \gamma_{35679} - i\,\gamma_{35689} + i\,\gamma_{45679} + \gamma_{45689} \\
&\quad -i\,\gamma_{123710} - \gamma_{123810} + \gamma_{124710} - i\,\gamma_{124810} - i\,\gamma_{356710} - \gamma_{356810} + \gamma_{456710} - i\,\gamma_{456810} \\
\Phi^6 &= +\phi_{12579} - i\,\phi_{12589} + i\,\phi_{12679} + \phi_{12689} + \phi_{34579} - i\,\phi_{34589} + i\,\phi_{34679} + \phi_{34689} \\
&\quad -i\,\phi_{125710} - \phi_{125810} + \phi_{126710} - i\,\phi_{126810} - i\,\phi_{345710} - \phi_{345810} + \phi_{346710} - i\,\phi_{346810} \\
\Gamma^6 &= +\gamma_{12579} - i\,\gamma_{12589} + i\,\gamma_{12679} + \gamma_{12689} + \gamma_{34579} - i\,\gamma_{34589} + i\,\gamma_{34679} + \gamma_{34689} \\
&\quad -i\,\gamma_{125710} - \gamma_{125810} + \gamma_{126710} - i\,\gamma_{126810} - i\,\gamma_{345710} - \gamma_{345810} + \gamma_{346710} - i\,\gamma_{346810} \\
\Phi^7 &= -i\,\phi_{13579} - \phi_{13589} + \phi_{13679} - i\,\phi_{13689} + \phi_{14579} - i\,\phi_{14589} + i\,\phi_{14679} + \phi_{14689} \\
&\quad +\phi_{23579} - i\,\phi_{23589} + i\,\phi_{23679} + \phi_{23689} + i\,\phi_{24579} + \phi_{24589} - \phi_{24679} + i\,\phi_{24689} \\
&\quad -\phi_{135710} + i\,\phi_{135810} - i\,\phi_{136710} - \phi_{136810} - i\,\phi_{145710} - \phi_{145810} + \phi_{146710} - i\,\phi_{146810} \\
&\quad -i\,\phi_{235710} - \phi_{235810} + \phi_{236710} - i\,\phi_{236810} + \phi_{245710} - i\,\phi_{245810} + i\,\phi_{246710} + \phi_{246810} \\
\Gamma^7 &= -i\,\gamma_{13579} - \gamma_{13589} + \gamma_{13679} - i\,\gamma_{13689} + \gamma_{14579} - i\,\gamma_{14589} + i\,\gamma_{14679} + \gamma_{14689} \\
&\quad +\gamma_{23579} - i\,\gamma_{23589} + i\,\gamma_{23679} + \gamma_{23689} + i\,\gamma_{24579} + \gamma_{24589} - \gamma_{24679} + i\,\gamma_{24689} \\
&\quad -\gamma_{135710} + i\,\gamma_{135810} - i\,\gamma_{136710} - \gamma_{136810} - i\,\gamma_{145710} - \gamma_{145810} + \gamma_{146710} - i\,\gamma_{146810} \\
&\quad -i\,\gamma_{235710} - \gamma_{235810} + \gamma_{236710} - i\,\gamma_{236810} + \gamma_{245710} - i\,\gamma_{245810} + i\,\gamma_{246710} + \gamma_{246810} \\
\Phi^8 &= -i\,\phi_{13579} - \phi_{13589} - \phi_{13679} + i\,\phi_{13689} - \phi_{14579} + i\,\phi_{14589} + i\,\phi_{14679} + \phi_{14689} \\
&\quad +\phi_{23579} - i\,\phi_{23589} - i\,\phi_{23679} - \phi_{23689} - i\,\phi_{24579} - \phi_{24589} - \phi_{24679} + i\,\phi_{24689} \\
&\quad -\phi_{135710} + i\,\phi_{135810} + i\,\phi_{136710} + \phi_{136810} + i\,\phi_{145710} + \phi_{145810} + \phi_{146710} - i\,\phi_{146810} \\
&\quad -i\,\phi_{235710} - \phi_{235810} - \phi_{236710} + i\,\phi_{236810} - \phi_{245710} + i\,\phi_{245810} + i\,\phi_{246710} + \phi_{246810} \\
\Gamma^8 &= -i\,\gamma_{13579} - \gamma_{13589} - \gamma_{13679} + i\,\gamma_{13689} - \gamma_{14579} + i\,\gamma_{14589} + i\,\gamma_{14679} + \gamma_{14689} \\
&\quad +\gamma_{23579} - i\,\gamma_{23589} - i\,\gamma_{23679} - \gamma_{23689} - i\,\gamma_{24579} - \gamma_{24589} - \gamma_{24679} + i\,\gamma_{24689} \\
&\quad -\gamma_{135710} + i\,\gamma_{135810} + i\,\gamma_{136710} + \gamma_{136810} + i\,\gamma_{145710} + \gamma_{145810} + \gamma_{146710} - i\,\gamma_{146810} \\
&\quad -i\,\gamma_{235710} - \gamma_{235810} - \gamma_{236710} + i\,\gamma_{236810} - \gamma_{245710} + i\,\gamma_{245810} + i\,\gamma_{246710} + \gamma_{246810} \\
\phi^9 &= -i\,\phi_{13579} - \phi_{13589} - \phi_{13679} + i\,\phi_{13689} + \phi_{14579} - i\,\phi_{14589} - i\,\phi_{14679} - \phi_{14689} \\
&\quad -\phi_{23579} + i\,\phi_{23589} + i\,\phi_{23679} + \phi_{23689} - i\,\phi_{24579} - \phi_{24589} - \phi_{24679} + i\,\phi_{24689} \\
&\quad -\phi_{135710} + i\,\phi_{135810} + i\,\phi_{136710} + \phi_{136810} - i\,\phi_{145710} - \phi_{145810} - \phi_{146710} + i\,\phi_{146810} \\
&\quad +i\,\phi_{235710} + \phi_{235810} + \phi_{236710} - i\,\phi_{236810} - \phi_{245710} + i\,\phi_{245810} + i\,\phi_{246710} + \phi_{246810} \\
\Gamma^9 &= -i\,\gamma_{13579} - \gamma_{13589} - \gamma_{13679} + i\,\gamma_{13689} + \gamma_{14579} - i\,\gamma_{14589} - i\,\gamma_{14679} - \gamma_{14689} \\
&\quad -\gamma_{23579} + i\,\gamma_{23589} + i\,\gamma_{23679} + \gamma_{23689} - i\,\gamma_{24579} - \gamma_{24589} - \gamma_{24679} + i\,\gamma_{24689} \\
&\quad -\gamma_{135710} + i\,\gamma_{135810} + i\,\gamma_{136710} + \gamma_{136810} - i\,\gamma_{145710} - \gamma_{145810} - \gamma_{146710} + i\,\gamma_{146810} \\
&\quad +i\,\gamma_{235710} + \gamma_{235810} + \gamma_{236710} - i\,\gamma_{236810} - \gamma_{245710} + i\,\gamma_{245810} + i\,\gamma_{246710} + \gamma_{246810} \\
\Phi^{10} &= -i\,\phi_{13579} - \phi_{13589} + \phi_{13679} - i\,\phi_{13689} - \phi_{14579} + i\,\phi_{14589} - i\,\phi_{14679} - \phi_{14689} \\
&\quad -\phi_{23579} + i\,\phi_{23589} - i\,\phi_{23679} - \phi_{23689} + i\,\phi_{24579} + \phi_{24589} - \phi_{24679} + i\,\phi_{24689} \\
&\quad -\phi_{135710} + i\,\phi_{135810} - i\,\phi_{136710} - \phi_{136810} + i\,\phi_{145710} + \phi_{145810} - \phi_{146710} + i\,\phi_{146810} \\
&\quad +i\,\phi_{235710} + \phi_{235810} - \phi_{236710} + i\,\phi_{236810} + \phi_{245710} - i\,\phi_{245810} + i\,\phi_{246710} + \phi_{246810} \\
\Gamma^{10} &= -i\,\gamma_{13579} - \gamma_{13589} + \gamma_{13679} - i\,\gamma_{13689} - \gamma_{14579} + i\,\gamma_{14589} - i\,\gamma_{14679} - \gamma_{14689} \\
&\quad -\gamma_{23579} + i\,\gamma_{23589} - i\,\gamma_{23679} - \gamma_{23689} + i\,\gamma_{24579} + \gamma_{24589} - \gamma_{24679} + i\,\gamma_{24689} \\
&\quad -\gamma_{135710} + i\,\gamma_{135810} - i\,\gamma_{136710} - \gamma_{136810} + i\,\gamma_{145710} + \gamma_{145810} - \gamma_{146710} + i\,\gamma_{146810} \\
&\quad +i\,\gamma_{235710} + \gamma_{235810} - \gamma_{236710} + i\,\gamma_{236810} + \gamma_{245710} - i\,\gamma_{245810} + i\,\gamma_{246710} + \gamma_{246810}
\end{aligned}
$$

The following complex scalar fields and lowering matrices make the $(\Delta = 0, 0)$ states $(3, 1, \bar{1}0)$ (see eq. 7.3)

$$
\begin{aligned}
\Phi^1 &= +i\,\phi_{13478} - i\,\phi_{15678} - \phi_{23478} + \phi_{25678} + i\,\phi_{134910} - i\,\phi_{156910} - \phi_{234910} + \phi_{256910} \\
\Gamma^1 &= -i\,\gamma_{13478} + i\,\gamma_{15678} - \gamma_{23478} + \gamma_{25678} - i\,\gamma_{134910} + i\,\gamma_{156910} - \gamma_{234910} + \gamma_{256910} \\
\Phi^2 &= -i\,\phi_{12378} + \phi_{12478} + i\,\phi_{35678} - \phi_{45678} - i\,\phi_{123910} + \phi_{124910} + i\,\phi_{356910} - \phi_{456910} \\
\Gamma^2 &= +i\,\gamma_{12378} + \gamma_{12478} - i\,\gamma_{35678} - \gamma_{45678} + i\,\gamma_{123910} + \gamma_{124910} - i\,\gamma_{356910} - \gamma_{456910} \\
\Phi^3 &= +i\,\phi_{12578} - \phi_{12678} - i\,\phi_{34578} + \phi_{34678} + i\,\phi_{125910} - \phi_{126910} - i\,\phi_{345910} + \phi_{346910} \\
\Gamma^3 &= -i\,\gamma_{12578} - \gamma_{12678} + i\,\gamma_{34578} + \gamma_{34678} - i\,\gamma_{125910} - \gamma_{126910} + i\,\gamma_{345910} + \gamma_{346910} \\
\Phi^4 &= -i\,\phi_{13478} - i\,\phi_{15678} + \phi_{23478} + \phi_{25678} - i\,\phi_{134910} - i\,\phi_{156910} + \phi_{234910} + \phi_{256910} \\
\Gamma^4 &= -i\,\gamma_{13478} - i\,\gamma_{15678} + \gamma_{23478} + \gamma_{25678} - i\,\gamma_{134910} - i\,\gamma_{156910} + \gamma_{234910} + \gamma_{256910} \\
\Phi^5 &= -i\,\phi_{12378} + \phi_{12478} - i\,\phi_{35678} + \phi_{45678} - i\,\phi_{123910} + \phi_{124910} - i\,\phi_{356910} + \phi_{456910} \\
\Gamma^5 &= -i\,\gamma_{12378} + \gamma_{12478} - i\,\gamma_{35678} + \gamma_{45678} - i\,\gamma_{123910} + \gamma_{124910} - i\,\gamma_{356910} + \gamma_{456910} \\
\Phi^6 &= -i\,\phi_{12578} - \phi_{12678} + i\,\phi_{34578} + \phi_{34678} + i\,\phi_{125910} + \phi_{126910} - i\,\phi_{345910} - \phi_{346910} \\
\Gamma^6 &= +i\,\gamma_{12578} - \gamma_{12678} - i\,\gamma_{34578} + \gamma_{34678} - i\,\gamma_{125910} + \gamma_{126910} + i\,\gamma_{345910} - \gamma_{346910} \\
\Phi^7 &= -\phi_{13578} + i\,\phi_{13678} + i\,\phi_{14578} + \phi_{14678} + i\,\phi_{23578} + \phi_{23678} + \phi_{24578} - i\,\phi_{24678} \\
&\quad -\phi_{135910} + i\,\phi_{136910} + i\,\phi_{145910} + \phi_{146910} + i\,\phi_{235910} + \phi_{236910} + \phi_{245910} - i\,\phi_{246910} \\
\Gamma^7 &= -\gamma_{13578} - i\,\gamma_{13678} - i\,\gamma_{14578} + \gamma_{14678} - i\,\gamma_{23578} + \gamma_{23678} + \gamma_{24578} + i\,\gamma_{24678} \\
&\quad -\gamma_{135910} - i\,\gamma_{136910} - i\,\gamma_{145910} + \gamma_{146910} - i\,\gamma_{235910} + \gamma_{236910} + \gamma_{245910} + i\,\gamma_{246910} \\
\Phi^8 &= -\phi_{13578} - i\,\phi_{13678} - i\,\phi_{14578} + \phi_{14678} + i\,\phi_{23578} - \phi_{23678} - \phi_{24578} - i\,\phi_{24678} \\
&\quad -\phi_{135910} + i\,\phi_{136910} + i\,\phi_{145910} + \phi_{146910} - i\,\phi_{235910} - \phi_{236910} - \phi_{245910} + i\,\phi_{246910} \\
\Gamma^8 &= -\gamma_{13578} + i\,\gamma_{13678} + i\,\gamma_{14578} + \gamma_{14678} - i\,\gamma_{23578} - \gamma_{23678} - \gamma_{24578} + i\,\gamma_{24678} \\
&\quad -\gamma_{135910} - i\,\gamma_{136910} - i\,\gamma_{145910} + \gamma_{146910} + i\,\gamma_{235910} - \gamma_{236910} - \gamma_{245910} - i\,\gamma_{246910}
\end{aligned}
$$

$$
\begin{aligned}
\Phi^9 &= -\phi_{13578} - i\,\phi_{13678} + i\,\phi_{14578} - \phi_{14678} - i\,\phi_{23578} + \phi_{23678} - \phi_{24578} - i\,\phi_{24678} \\
&\quad -\phi_{135910} - i\,\phi_{136910} + i\,\phi_{145910} - \phi_{146910} - i\,\phi_{235910} + \phi_{236910} - \phi_{245910} - i\,\phi_{246910} \\
\Gamma^9 &= -\gamma_{13578} + i\,\gamma_{13678} - i\,\gamma_{14578} - \gamma_{14678} + i\,\gamma_{23578} + \gamma_{23678} - \gamma_{24578} + i\,\gamma_{24678} \\
&\quad -\gamma_{135910} + i\,\gamma_{136910} - i\,\gamma_{145910} - \gamma_{146910} + i\,\gamma_{235910} + \gamma_{236910} - \gamma_{245910} + i\,\gamma_{246910} \\
\Phi^{10} &= -\phi_{13578} + i\,\phi_{13678} - i\,\phi_{14578} - \phi_{14678} - i\,\phi_{23578} - \phi_{23678} + \phi_{24578} - i\,\phi_{24678} \\
&\quad -\phi_{135910} + i\,\phi_{136910} - i\,\phi_{145910} - \phi_{146910} - i\,\phi_{235910} - \phi_{236910} + \phi_{245910} - i\,\phi_{246910} \\
\Gamma^{10} &= -\gamma_{13578} - i\,\gamma_{13678} + i\,\gamma_{14578} - \gamma_{14678} + i\,\gamma_{23578} - \gamma_{23678} + \gamma_{24578} + i\,\gamma_{24678} \\
&\quad -\gamma_{135910} - i\,\gamma_{136910} + i\,\gamma_{145910} - \gamma_{146910} + i\,\gamma_{235910} - \gamma_{236910} + \gamma_{245910} + i\,\gamma_{246910}
\end{aligned}
$$

The following complex scalar fields and lowering matrices make the $(\downarrow, 0)$ states $(3, 1, \bar{10})$ (see eq. 7.3)

$$
\begin{aligned}
\Phi^1 &= -\phi_{13479} - i\,\phi_{13489} + \phi_{15679} + i\,\phi_{15689} + i\,\phi_{23479} - \phi_{23489} - i\,\phi_{25679} + \phi_{25689} \\
&\quad -i\,\phi_{134710} + \phi_{134810} + i\,\phi_{156710} - \phi_{156810} - \phi_{234710} - i\,\phi_{234810} + \phi_{256710} + i\,\phi_{256810} \\
\Gamma^1 &= -\gamma_{13479} - i\,\gamma_{13489} + \gamma_{15679} + i\,\gamma_{15689} + i\,\gamma_{23479} - \gamma_{23489} - i\,\gamma_{25679} + \gamma_{25689} \\
&\quad -i\,\gamma_{134710} + \gamma_{134810} + i\,\gamma_{156710} - \gamma_{156810} - \gamma_{234710} - i\,\gamma_{234810} + \gamma_{256710} + i\,\gamma_{256810} \\
\Phi^2 &= +\phi_{12379} + i\,\phi_{12389} - i\,\phi_{12479} + \phi_{12489} - \phi_{35679} - i\,\phi_{35689} + i\,\phi_{45679} - \phi_{45689} \\
&\quad +i\,\phi_{123710} - \phi_{123810} + \phi_{124710} + i\,\phi_{124810} - i\,\phi_{356710} + \phi_{356810} - \phi_{456710} - i\,\phi_{456810} \\
\Gamma^2 &= +\gamma_{12379} + i\,\gamma_{12389} - i\,\gamma_{12479} + \gamma_{12489} - \gamma_{35679} - i\,\gamma_{35689} + i\,\gamma_{45679} - \gamma_{45689} \\
&\quad +i\,\gamma_{123710} - \gamma_{123810} + \gamma_{124710} + i\,\gamma_{124810} - i\,\gamma_{356710} + \gamma_{356810} - \gamma_{456710} - i\,\gamma_{456810} \\
\Phi^3 &= -\phi_{12579} - i\,\phi_{12589} + i\,\phi_{12679} - \phi_{12689} + \phi_{34579} + i\,\phi_{34589} - i\,\phi_{34679} + \phi_{34689} \\
&\quad -i\,\phi_{125710} + \phi_{125810} - \phi_{126710} - i\,\phi_{126810} + i\,\phi_{345710} - \phi_{345810} + \phi_{346710} + i\,\phi_{346810} \\
\Gamma^3 &= -\gamma_{12579} - i\,\gamma_{12589} + i\,\gamma_{12679} - \gamma_{12689} + \gamma_{34579} + i\,\gamma_{34589} - i\,\gamma_{34679} + \gamma_{34689} \\
&\quad -i\,\gamma_{125710} + \gamma_{125810} - \gamma_{126710} - i\,\gamma_{126810} + i\,\gamma_{345710} - \gamma_{345810} + \gamma_{346710} + i\,\gamma_{346810} \\
\Phi^4 &= -\phi_{13479} - i\,\phi_{13489} - \phi_{15679} - i\,\phi_{15689} - i\,\phi_{23479} + \phi_{23489} - i\,\phi_{25679} + \phi_{25689} \\
&\quad -i\,\phi_{134710} + \phi_{134810} - i\,\phi_{156710} + \phi_{156810} + \phi_{234710} + i\,\phi_{234810} + \phi_{256710} + i\,\phi_{256810} \\
\Gamma^4 &= -\gamma_{13479} - i\,\gamma_{13489} - \gamma_{15679} - i\,\gamma_{15689} - i\,\gamma_{23479} + \gamma_{23489} - i\,\gamma_{25679} + \gamma_{25689} \\
&\quad -i\,\gamma_{134710} + \gamma_{134810} - i\,\gamma_{156710} + \gamma_{156810} + \gamma_{234710} + i\,\gamma_{234810} + \gamma_{256710} + i\,\gamma_{256810} \\
\Phi^5 &= -\phi_{12379} - i\,\phi_{12389} - i\,\phi_{12479} + \phi_{12489} - \phi_{35679} - i\,\phi_{35689} - i\,\phi_{45679} + \phi_{45689} \\
&\quad -i\,\phi_{123710} + \phi_{123810} + \phi_{124710} + i\,\phi_{124810} - i\,\phi_{356710} + \phi_{356810} + \phi_{456710} + i\,\phi_{456810} \\
\Gamma^5 &= -\gamma_{12379} - i\,\gamma_{12389} - i\,\gamma_{12479} + \gamma_{12489} - \gamma_{35679} - i\,\gamma_{35689} - i\,\gamma_{45679} + \gamma_{45689} \\
&\quad -i\,\gamma_{123710} + \gamma_{123810} + \gamma_{124710} + i\,\gamma_{124810} - i\,\gamma_{356710} + \gamma_{356810} + \gamma_{456710} + i\,\gamma_{456810} \\
\Phi^6 &= -\phi_{12579} - i\,\phi_{12589} - i\,\phi_{12679} + \phi_{12689} - \phi_{34579} - i\,\phi_{34589} - i\,\phi_{34679} + \phi_{34689} \\
&\quad -i\,\phi_{125710} + \phi_{125810} + \phi_{126710} + i\,\phi_{126810} - i\,\phi_{345710} + \phi_{345810} + \phi_{346710} + i\,\phi_{346810} \\
\Gamma^6 &= -\gamma_{12579} - i\,\gamma_{12589} - i\,\gamma_{12679} + \gamma_{12689} - \gamma_{34579} - i\,\gamma_{34589} - i\,\gamma_{34679} + \gamma_{34689} \\
&\quad -i\,\gamma_{125710} + \gamma_{125810} + \gamma_{126710} + i\,\gamma_{126810} - i\,\gamma_{345710} + \gamma_{345810} + \gamma_{346710} + i\,\gamma_{346810} \\
\Phi^7 &= +i\,\phi_{13579} - \phi_{13589} - \phi_{13679} - i\,\phi_{13689} - \phi_{14579} - i\,\phi_{14589} - i\,\phi_{14679} + \phi_{14689} \\
&\quad -\phi_{23579} - i\,\phi_{23589} - i\,\phi_{23679} + \phi_{23689} - i\,\phi_{24579} + \phi_{24589} + \phi_{24679} + i\,\phi_{24689} \\
&\quad -\phi_{135710} - i\,\phi_{135810} - i\,\phi_{136710} + \phi_{136810} - i\,\phi_{145710} + \phi_{145810} + \phi_{146710} + i\,\phi_{146810} \\
&\quad -i\,\phi_{235710} + \phi_{235810} + \phi_{236710} + i\,\phi_{236810} + \phi_{245710} + i\,\phi_{245810} + i\,\phi_{246710} - \phi_{246810} \\
\Gamma^7 &= +i\,\gamma_{13579} - \gamma_{13589} - \gamma_{13679} - i\,\gamma_{13689} - \gamma_{14579} - i\,\gamma_{14589} - i\,\gamma_{14679} + \gamma_{14689} \\
&\quad -\gamma_{23579} - i\,\gamma_{23589} - i\,\gamma_{23679} + \gamma_{23689} - i\,\gamma_{24579} + \gamma_{24589} + \gamma_{24679} + i\,\gamma_{24689} \\
&\quad -\gamma_{135710} - i\,\gamma_{135810} - i\,\gamma_{136710} + \gamma_{136810} - i\,\gamma_{145710} + \gamma_{145810} + \gamma_{146710} + i\,\gamma_{146810} \\
&\quad -i\,\gamma_{235710} + \gamma_{235810} + \gamma_{236710} + i\,\gamma_{236810} + \gamma_{245710} + i\,\gamma_{245810} + i\,\gamma_{246710} - \gamma_{246810} \\
\Phi^8 &= +i\,\phi_{13579} - \phi_{13589} + \phi_{13679} + i\,\phi_{13689} + \phi_{14579} + i\,\phi_{14589} - i\,\phi_{14679} + \phi_{14689} \\
&\quad -\phi_{23579} - i\,\phi_{23589} + i\,\phi_{23679} - \phi_{23689} + i\,\phi_{24579} - \phi_{24589} + \phi_{24679} + i\,\phi_{24689} \\
&\quad -\phi_{135710} - i\,\phi_{135810} + i\,\phi_{136710} - \phi_{136810} + i\,\phi_{145710} - \phi_{145810} + \phi_{146710} + i\,\phi_{146810} \\
&\quad -i\,\phi_{235710} + \phi_{235810} - \phi_{236710} - i\,\phi_{236810} - \phi_{245710} - i\,\phi_{245810} + i\,\phi_{246710} - \phi_{246810} \\
\Gamma^8 &= +i\,\gamma_{13579} - \gamma_{13589} + \gamma_{13679} + i\,\gamma_{13689} + \gamma_{14579} + i\,\gamma_{14589} - i\,\gamma_{14679} + \gamma_{14689} \\
&\quad -\gamma_{23579} - i\,\gamma_{23589} + i\,\gamma_{23679} - \gamma_{23689} + i\,\gamma_{24579} - \gamma_{24589} + \gamma_{24679} + i\,\gamma_{24689} \\
&\quad -\gamma_{135710} - i\,\gamma_{135810} + i\,\gamma_{136710} - \gamma_{136810} + i\,\gamma_{145710} - \gamma_{145810} + \gamma_{146710} + i\,\gamma_{146810} \\
&\quad -i\,\gamma_{235710} + \gamma_{235810} - \gamma_{236710} - i\,\gamma_{236810} - \gamma_{245710} - i\,\gamma_{245810} + i\,\gamma_{246710} - \gamma_{246810} \\
\Phi^9 &= +i\,\phi_{13579} - \phi_{13589} + \phi_{13679} + i\,\phi_{13689} - \phi_{14579} - i\,\phi_{14589} + i\,\phi_{14679} - \phi_{14689} \\
&\quad +\phi_{23579} + i\,\phi_{23589} - i\,\phi_{23679} + \phi_{23689} + i\,\phi_{24579} - \phi_{24589} + \phi_{24679} + i\,\phi_{24689} \\
&\quad -\phi_{135710} - i\,\phi_{135810} + i\,\phi_{136710} - \phi_{136810} - i\,\phi_{145710} + \phi_{145810} - \phi_{146710} - i\,\phi_{146810} \\
&\quad +i\,\phi_{235710} - \phi_{235810} + \phi_{236710} + i\,\phi_{236810} - \phi_{245710} - i\,\phi_{245810} + i\,\phi_{246710} - \phi_{246810} \\
\Gamma^9 &= +i\,\gamma_{13579} - \gamma_{13589} + \gamma_{13679} + i\,\gamma_{13689} - \gamma_{14579} - i\,\gamma_{14589} + i\,\gamma_{14679} - \gamma_{14689} \\
&\quad +\gamma_{23579} + i\,\gamma_{23589} - i\,\gamma_{23679} + \gamma_{23689} + i\,\gamma_{24579} - \gamma_{24589} + \gamma_{24679} + i\,\gamma_{24689} \\
&\quad -\gamma_{135710} - i\,\gamma_{135810} + i\,\gamma_{136710} - \gamma_{136810} - i\,\gamma_{145710} + \gamma_{145810} - \gamma_{146710} - i\,\gamma_{146810} \\
&\quad +i\,\gamma_{235710} - \gamma_{235810} + \gamma_{236710} + i\,\gamma_{236810} - \gamma_{245710} - i\,\gamma_{245810} + i\,\gamma_{246710} - \gamma_{246810} \\
\Phi^{10} &= +i\,\phi_{13579} - \phi_{13589} - \phi_{13679} - i\,\phi_{13689} + \phi_{14579} + i\,\phi_{14589} + i\,\phi_{14679} - \phi_{14689} \\
&\quad +\phi_{23579} + i\,\phi_{23589} + i\,\phi_{23679} - \phi_{23689} - i\,\phi_{24579} + \phi_{24589} + \phi_{24679} + i\,\phi_{24689} \\
&\quad -\phi_{135710} - i\,\phi_{135810} - i\,\phi_{136710} + \phi_{136810} + i\,\phi_{145710} - \phi_{145810} - \phi_{146710} - i\,\phi_{146810} \\
&\quad +i\,\phi_{235710} - \phi_{235810} - \phi_{236710} - i\,\phi_{236810} + \phi_{245710} + i\,\phi_{245810} + i\,\phi_{246710} - \phi_{246810} \\
\Gamma^{10} &= +i\,\gamma_{13579} - \gamma_{13589} - \gamma_{13679} - i\,\gamma_{13689} + \gamma_{14579} + i\,\gamma_{14589} + i\,\gamma_{14679} - \gamma_{14689} \\
&\quad +\gamma_{23579} + i\,\gamma_{23589} + i\,\gamma_{23679} - \gamma_{23689} - i\,\gamma_{24579} + \gamma_{24589} + \gamma_{24679} + i\,\gamma_{24689} \\
&\quad -\gamma_{135710} - i\,\gamma_{135810} - i\,\gamma_{136710} + \gamma_{136810} + i\,\gamma_{145710} - \gamma_{145810} - \gamma_{146710} - i\,\gamma_{146810} \\
&\quad +i\,\gamma_{235710} - \gamma_{235810} - \gamma_{236710} - i\,\gamma_{236810} + \gamma_{245710} + i\,\gamma_{245810} + i\,\gamma_{246710} - \gamma_{246810}
\end{aligned}
$$

The following Complex scalar fields and lowering matrices make the $(0, 0)$ states of $(1, 1, 6)$. (see eq. 7.3)

$$
\begin{aligned}
\Phi^1 &= -i\,\phi_{13456} + \phi_{23456} + i\,\phi_{178910} - \phi_{278910} \\
\Gamma^1 &= +i\,\gamma_{13456} + \gamma_{23456} - i\,\gamma_{178910} - \gamma_{278910} \\
\Phi^2 &= +i\,\phi_{12356} - \phi_{12456} - i\,\phi_{378910} + \phi_{478910} \\
\Gamma^2 &= -i\,\gamma_{12356} - \gamma_{12456} + i\,\gamma_{378910} + \gamma_{478910} \\
\Phi^3 &= -i\,\phi_{12345} + \phi_{21346} + i\,\phi_{578910} - \phi_{678910} \\
\Gamma^3 &= +i\,\gamma_{12345} + \gamma_{21346} - i\,\gamma_{578910} - \gamma_{678910} \\
\Phi^4 &= +i\,\phi_{13456} + \phi_{23456} + i\,\phi_{178910} + \phi_{278910} \\
\Gamma^4 &= -i\,\gamma_{13456} + \gamma_{23456} - i\,\gamma_{178910} + \gamma_{278910} \\
\Phi^5 &= -i\,\phi_{12356} - \phi_{12456} - i\,\phi_{378910} - \phi_{478910} \\
\Gamma^5 &= +i\,\gamma_{12356} - \gamma_{12456} + i\,\gamma_{378910} - \gamma_{478910} \\
\Phi^6 &= +i\,\phi_{12345} + \phi_{12346} + i\,\phi_{578910} + \phi_{678910} \\
\Gamma^6 &= -i\,\gamma_{12345} + \gamma_{12346} - i\,\gamma_{578910} + \gamma_{678910}
\end{aligned}
$$

C. CHARTS FOR WEIGHTS OF L_3, R_3, U_{3-8-15} AND Q

In the following pages, we present for basis C, the eigenvalues at the sites of 32×32 matrices in the adjoint representation. We proceed in the order L_3 , R_3 , U_3, U_8, U_{15} and Q. Each one is divided into 4 blocks which are given in the order (11), (21), (12), (22) respectively. They are obtained as explained in Appendix A in eqs. A.1 to A.3.

$$\begin{bmatrix}
0 & 0 & 0 & 0 & -1 & -1 & -1 & -1 & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\
0 & 0 & 0 & 0 & -1 & -1 & -1 & -1 & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\
0 & 0 & 0 & 0 & -1 & -1 & -1 & -1 & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\
0 & 0 & 0 & 0 & -1 & -1 & -1 & -1 & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\
1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\
1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\
1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\
1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\
\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}_{11}$$

$$\begin{bmatrix}
\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & -1 & -1 & -1 & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\
0 & 0 & 0 & 0 & -1 & -1 & -1 & -1 & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\
0 & 0 & 0 & 0 & -1 & -1 & -1 & -1 & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\
0 & 0 & 0 & 0 & -1 & -1 & -1 & -1 & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\
1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\
1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\
1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\
1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2}
\end{bmatrix}_{21}$$

$$
\begin{bmatrix}
-\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & 0 & 0 & 0 & 0 & -1 & -1 & -1 & -1 \\
-\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & 0 & 0 & 0 & 0 & -1 & -1 & -1 & -1 \\
-\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & 0 & 0 & 0 & 0 & -1 & -1 & -1 & -1 \\
-\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & 0 & 0 & 0 & 0 & -1 & -1 & -1 & -1 \\
\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\
\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\
\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\
\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2}
\end{bmatrix}_{12}
$$

$$
\begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\
-\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & 0 & 0 & 0 & 0 & -1 & -1 & -1 & -1 \\
-\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & 0 & 0 & 0 & 0 & -1 & -1 & -1 & -1 \\
-\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & 0 & 0 & 0 & 0 & -1 & -1 & -1 & -1 \\
-\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & 0 & 0 & 0 & 0 & -1 & -1 & -1 & -1 \\
\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\
\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\
\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\
\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0
\end{bmatrix}_{22}
$$

$$
\begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\
-\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & 0 & 0 & 0 & 0 & -1 & -1 & -1 & -1 \\
-\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & 0 & 0 & 0 & 0 & -1 & -1 & -1 & -1 \\
-\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & 0 & 0 & 0 & 0 & -1 & -1 & -1 & -1 \\
-\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & 0 & 0 & 0 & 0 & -1 & -1 & -1 & -1 \\
\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\
\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\
\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\
\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0
\end{bmatrix}_{11}
$$

$$
\begin{bmatrix}
-\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & 0 & 0 & 0 & 0 & -1 & -1 & -1 & -1 \\
-\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & 0 & 0 & 0 & 0 & -1 & -1 & -1 & -1 \\
-\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & 0 & 0 & 0 & 0 & -1 & -1 & -1 & -1 \\
-\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & 0 & 0 & 0 & 0 & -1 & -1 & -1 & -1 \\
\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\
\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\
\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\
\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2}
\end{bmatrix}_{21}
$$

$$
\begin{bmatrix}
\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & -1 & -1 & -1 & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\
0 & 0 & 0 & 0 & -1 & -1 & -1 & -1 & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\
0 & 0 & 0 & 0 & -1 & -1 & -1 & -1 & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\
0 & 0 & 0 & 0 & -1 & -1 & -1 & -1 & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\
1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\
1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\
1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\
1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2}
\end{bmatrix}_{12}
$$

$$
\begin{bmatrix}
0 & 0 & 0 & 0 & -1 & -1 & -1 & -1 & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\
0 & 0 & 0 & 0 & -1 & -1 & -1 & -1 & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\
0 & 0 & 0 & 0 & -1 & -1 & -1 & -1 & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\
0 & 0 & 0 & 0 & -1 & -1 & -1 & -1 & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\
1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\
1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\
1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\
1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\
\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}_{22}
$$

$$\begin{bmatrix} 0 & -1 & -\frac{1}{2} & -\frac{1}{2} & 0 & -1 & -\frac{1}{2} & -\frac{1}{2} & -1 & 0 & -\frac{1}{2} & -\frac{1}{2} & -1 & 0 & -\frac{1}{2} & -\frac{1}{2} \\ 1 & 0 & \frac{1}{2} & \frac{1}{2} & 1 & 0 & \frac{1}{2} & \frac{1}{2} & 0 & 1 & \frac{1}{2} & \frac{1}{2} & 0 & 1 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & 0 & 0 & \frac{1}{2} & -\frac{1}{2} & 0 & 0 & -\frac{1}{2} & \frac{1}{2} & 0 & 0 & -\frac{1}{2} & \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & -\frac{1}{2} & 0 & 0 & \frac{1}{2} & -\frac{1}{2} & 0 & 0 & -\frac{1}{2} & \frac{1}{2} & 0 & 0 & -\frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & -1 & -\frac{1}{2} & -\frac{1}{2} & 0 & -1 & -\frac{1}{2} & -\frac{1}{2} & -1 & 0 & -\frac{1}{2} & -\frac{1}{2} & -1 & 0 & -\frac{1}{2} & -\frac{1}{2} \\ 1 & 0 & \frac{1}{2} & \frac{1}{2} & 1 & 0 & \frac{1}{2} & \frac{1}{2} & 0 & 1 & \frac{1}{2} & \frac{1}{2} & 0 & 1 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & 0 & 0 & \frac{1}{2} & -\frac{1}{2} & 0 & 0 & -\frac{1}{2} & \frac{1}{2} & 0 & 0 & -\frac{1}{2} & \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & -\frac{1}{2} & 0 & 0 & \frac{1}{2} & -\frac{1}{2} & 0 & 0 & -\frac{1}{2} & \frac{1}{2} & 0 & 0 & -\frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 1 & 0 & \frac{1}{2} & \frac{1}{2} & 1 & 0 & \frac{1}{2} & \frac{1}{2} & 0 & 1 & \frac{1}{2} & \frac{1}{2} & 0 & 1 & \frac{1}{2} & \frac{1}{2} \\ 0 & -1 & -\frac{1}{2} & -\frac{1}{2} & 0 & -1 & -\frac{1}{2} & -\frac{1}{2} & -1 & 0 & -\frac{1}{2} & -\frac{1}{2} & -1 & 0 & -\frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & 0 & 0 & \frac{1}{2} & -\frac{1}{2} & 0 & 0 & -\frac{1}{2} & \frac{1}{2} & 0 & 0 & -\frac{1}{2} & \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & -\frac{1}{2} & 0 & 0 & \frac{1}{2} & -\frac{1}{2} & 0 & 0 & -\frac{1}{2} & \frac{1}{2} & 0 & 0 & -\frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 1 & 0 & \frac{1}{2} & \frac{1}{2} & 1 & 0 & \frac{1}{2} & \frac{1}{2} & 0 & 1 & \frac{1}{2} & \frac{1}{2} & 0 & 1 & \frac{1}{2} & \frac{1}{2} \\ 0 & -1 & -\frac{1}{2} & -\frac{1}{2} & 0 & -1 & -\frac{1}{2} & -\frac{1}{2} & -1 & 0 & -\frac{1}{2} & -\frac{1}{2} & -1 & 0 & -\frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & 0 & 0 & \frac{1}{2} & -\frac{1}{2} & 0 & 0 & -\frac{1}{2} & \frac{1}{2} & 0 & 0 & -\frac{1}{2} & \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & -\frac{1}{2} & 0 & 0 & \frac{1}{2} & -\frac{1}{2} & 0 & 0 & -\frac{1}{2} & \frac{1}{2} & 0 & 0 & -\frac{1}{2} & \frac{1}{2} & 0 & 0 \end{bmatrix}_{11}$$

$$\begin{bmatrix} 0 & -1 & -\frac{1}{2} & -\frac{1}{2} & 0 & -1 & -\frac{1}{2} & -\frac{1}{2} & -1 & 0 & -\frac{1}{2} & -\frac{1}{2} & -1 & 0 & -\frac{1}{2} & -\frac{1}{2} \\ 1 & 0 & \frac{1}{2} & \frac{1}{2} & 1 & 0 & \frac{1}{2} & \frac{1}{2} & 0 & 1 & \frac{1}{2} & \frac{1}{2} & 0 & 1 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & 0 & 0 & \frac{1}{2} & -\frac{1}{2} & 0 & 0 & -\frac{1}{2} & \frac{1}{2} & 0 & 0 & -\frac{1}{2} & \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & -\frac{1}{2} & 0 & 0 & \frac{1}{2} & -\frac{1}{2} & 0 & 0 & -\frac{1}{2} & \frac{1}{2} & 0 & 0 & -\frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & -1 & -\frac{1}{2} & -\frac{1}{2} & 0 & -1 & -\frac{1}{2} & -\frac{1}{2} & -1 & 0 & -\frac{1}{2} & -\frac{1}{2} & -1 & 0 & -\frac{1}{2} & -\frac{1}{2} \\ 1 & 0 & \frac{1}{2} & \frac{1}{2} & 1 & 0 & \frac{1}{2} & \frac{1}{2} & 0 & 1 & \frac{1}{2} & \frac{1}{2} & 0 & 1 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & 0 & 0 & \frac{1}{2} & -\frac{1}{2} & 0 & 0 & -\frac{1}{2} & \frac{1}{2} & 0 & 0 & -\frac{1}{2} & \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & -\frac{1}{2} & 0 & 0 & \frac{1}{2} & -\frac{1}{2} & 0 & 0 & -\frac{1}{2} & \frac{1}{2} & 0 & 0 & -\frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 1 & 0 & \frac{1}{2} & \frac{1}{2} & 1 & 0 & \frac{1}{2} & \frac{1}{2} & 0 & 1 & \frac{1}{2} & \frac{1}{2} & 0 & 1 & \frac{1}{2} & \frac{1}{2} \\ 0 & -1 & -\frac{1}{2} & -\frac{1}{2} & 0 & -1 & -\frac{1}{2} & -\frac{1}{2} & -1 & 0 & -\frac{1}{2} & -\frac{1}{2} & -1 & 0 & -\frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & 0 & 0 & \frac{1}{2} & -\frac{1}{2} & 0 & 0 & -\frac{1}{2} & \frac{1}{2} & 0 & 0 & -\frac{1}{2} & \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & -\frac{1}{2} & 0 & 0 & \frac{1}{2} & -\frac{1}{2} & 0 & 0 & -\frac{1}{2} & \frac{1}{2} & 0 & 0 & -\frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 1 & 0 & \frac{1}{2} & \frac{1}{2} & 1 & 0 & \frac{1}{2} & \frac{1}{2} & 0 & 1 & \frac{1}{2} & \frac{1}{2} & 0 & 1 & \frac{1}{2} & \frac{1}{2} \\ 0 & -1 & -\frac{1}{2} & -\frac{1}{2} & 0 & -1 & -\frac{1}{2} & -\frac{1}{2} & -1 & 0 & -\frac{1}{2} & -\frac{1}{2} & -1 & 0 & -\frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & 0 & 0 & \frac{1}{2} & -\frac{1}{2} & 0 & 0 & -\frac{1}{2} & \frac{1}{2} & 0 & 0 & -\frac{1}{2} & \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & -\frac{1}{2} & 0 & 0 & \frac{1}{2} & -\frac{1}{2} & 0 & 0 & -\frac{1}{2} & \frac{1}{2} & 0 & 0 & -\frac{1}{2} & \frac{1}{2} & 0 & 0 \end{bmatrix}_{21}$$

$$\begin{bmatrix}
0 & -1 & -\frac{1}{2} & -\frac{1}{2} & 0 & -1 & -\frac{1}{2} & -\frac{1}{2} & -1 & 0 & -\frac{1}{2} & -\frac{1}{2} & -1 & 0 & -\frac{1}{2} & -\frac{1}{2} \\
1 & 0 & \frac{1}{2} & \frac{1}{2} & 1 & 0 & \frac{1}{2} & \frac{1}{2} & 0 & 1 & \frac{1}{2} & \frac{1}{2} & 0 & 1 & \frac{1}{2} & \frac{1}{2} \\
\frac{1}{2} & -\frac{1}{2} & 0 & 0 & \frac{1}{2} & -\frac{1}{2} & 0 & 0 & -\frac{1}{2} & \frac{1}{2} & 0 & 0 & -\frac{1}{2} & \frac{1}{2} & 0 & 0 \\
\frac{1}{2} & -\frac{1}{2} & 0 & 0 & \frac{1}{2} & -\frac{1}{2} & 0 & 0 & -\frac{1}{2} & \frac{1}{2} & 0 & 0 & -\frac{1}{2} & \frac{1}{2} & 0 & 0 \\
0 & -1 & -\frac{1}{2} & -\frac{1}{2} & 0 & -1 & -\frac{1}{2} & -\frac{1}{2} & -1 & 0 & -\frac{1}{2} & -\frac{1}{2} & -1 & 0 & -\frac{1}{2} & -\frac{1}{2} \\
1 & 0 & \frac{1}{2} & \frac{1}{2} & 1 & 0 & \frac{1}{2} & \frac{1}{2} & 0 & 1 & \frac{1}{2} & \frac{1}{2} & 0 & 1 & \frac{1}{2} & \frac{1}{2} \\
\frac{1}{2} & -\frac{1}{2} & 0 & 0 & \frac{1}{2} & -\frac{1}{2} & 0 & 0 & -\frac{1}{2} & \frac{1}{2} & 0 & 0 & -\frac{1}{2} & \frac{1}{2} & 0 & 0 \\
\frac{1}{2} & -\frac{1}{2} & 0 & 0 & \frac{1}{2} & -\frac{1}{2} & 0 & 0 & -\frac{1}{2} & \frac{1}{2} & 0 & 0 & -\frac{1}{2} & \frac{1}{2} & 0 & 0 \\
1 & 0 & \frac{1}{2} & \frac{1}{2} & 1 & 0 & \frac{1}{2} & \frac{1}{2} & 0 & 1 & \frac{1}{2} & \frac{1}{2} & 0 & 1 & \frac{1}{2} & \frac{1}{2} \\
0 & -1 & -\frac{1}{2} & -\frac{1}{2} & 0 & -1 & -\frac{1}{2} & -\frac{1}{2} & -1 & 0 & -\frac{1}{2} & -\frac{1}{2} & -1 & 0 & -\frac{1}{2} & -\frac{1}{2} \\
\frac{1}{2} & -\frac{1}{2} & 0 & 0 & \frac{1}{2} & -\frac{1}{2} & 0 & 0 & -\frac{1}{2} & \frac{1}{2} & 0 & 0 & -\frac{1}{2} & \frac{1}{2} & 0 & 0 \\
\frac{1}{2} & -\frac{1}{2} & 0 & 0 & \frac{1}{2} & -\frac{1}{2} & 0 & 0 & -\frac{1}{2} & \frac{1}{2} & 0 & 0 & -\frac{1}{2} & \frac{1}{2} & 0 & 0 \\
1 & 0 & \frac{1}{2} & \frac{1}{2} & 1 & 0 & \frac{1}{2} & \frac{1}{2} & 0 & 1 & \frac{1}{2} & \frac{1}{2} & 0 & 1 & \frac{1}{2} & \frac{1}{2} \\
0 & -1 & -\frac{1}{2} & -\frac{1}{2} & 0 & -1 & -\frac{1}{2} & -\frac{1}{2} & -1 & 0 & -\frac{1}{2} & -\frac{1}{2} & -1 & 0 & -\frac{1}{2} & -\frac{1}{2} \\
\frac{1}{2} & -\frac{1}{2} & 0 & 0 & \frac{1}{2} & -\frac{1}{2} & 0 & 0 & -\frac{1}{2} & \frac{1}{2} & 0 & 0 & -\frac{1}{2} & \frac{1}{2} & 0 & 0 \\
\frac{1}{2} & -\frac{1}{2} & 0 & 0 & \frac{1}{2} & -\frac{1}{2} & 0 & 0 & -\frac{1}{2} & \frac{1}{2} & 0 & 0 & -\frac{1}{2} & \frac{1}{2} & 0 & 0
\end{bmatrix}_{12}$$

$$\begin{bmatrix}
0 & -1 & -\frac{1}{2} & -\frac{1}{2} & 0 & -1 & -\frac{1}{2} & -\frac{1}{2} & -1 & 0 & -\frac{1}{2} & -\frac{1}{2} & -1 & 0 & -\frac{1}{2} & -\frac{1}{2} \\
1 & 0 & \frac{1}{2} & \frac{1}{2} & 1 & 0 & \frac{1}{2} & \frac{1}{2} & 0 & 1 & \frac{1}{2} & \frac{1}{2} & 0 & 1 & \frac{1}{2} & \frac{1}{2} \\
\frac{1}{2} & -\frac{1}{2} & 0 & 0 & \frac{1}{2} & -\frac{1}{2} & 0 & 0 & -\frac{1}{2} & \frac{1}{2} & 0 & 0 & -\frac{1}{2} & \frac{1}{2} & 0 & 0 \\
\frac{1}{2} & -\frac{1}{2} & 0 & 0 & \frac{1}{2} & -\frac{1}{2} & 0 & 0 & -\frac{1}{2} & \frac{1}{2} & 0 & 0 & -\frac{1}{2} & \frac{1}{2} & 0 & 0 \\
0 & -1 & -\frac{1}{2} & -\frac{1}{2} & 0 & -1 & -\frac{1}{2} & -\frac{1}{2} & -1 & 0 & -\frac{1}{2} & -\frac{1}{2} & -1 & 0 & -\frac{1}{2} & -\frac{1}{2} \\
1 & 0 & \frac{1}{2} & \frac{1}{2} & 1 & 0 & \frac{1}{2} & \frac{1}{2} & 0 & 1 & \frac{1}{2} & \frac{1}{2} & 0 & 1 & \frac{1}{2} & \frac{1}{2} \\
\frac{1}{2} & -\frac{1}{2} & 0 & 0 & \frac{1}{2} & -\frac{1}{2} & 0 & 0 & -\frac{1}{2} & \frac{1}{2} & 0 & 0 & -\frac{1}{2} & \frac{1}{2} & 0 & 0 \\
\frac{1}{2} & -\frac{1}{2} & 0 & 0 & \frac{1}{2} & -\frac{1}{2} & 0 & 0 & -\frac{1}{2} & \frac{1}{2} & 0 & 0 & -\frac{1}{2} & \frac{1}{2} & 0 & 0 \\
1 & 0 & \frac{1}{2} & \frac{1}{2} & 1 & 0 & \frac{1}{2} & \frac{1}{2} & 0 & 1 & \frac{1}{2} & \frac{1}{2} & 0 & 1 & \frac{1}{2} & \frac{1}{2} \\
0 & -1 & -\frac{1}{2} & -\frac{1}{2} & 0 & -1 & -\frac{1}{2} & -\frac{1}{2} & -1 & 0 & -\frac{1}{2} & -\frac{1}{2} & -1 & 0 & -\frac{1}{2} & -\frac{1}{2} \\
\frac{1}{2} & -\frac{1}{2} & 0 & 0 & \frac{1}{2} & -\frac{1}{2} & 0 & 0 & -\frac{1}{2} & \frac{1}{2} & 0 & 0 & -\frac{1}{2} & \frac{1}{2} & 0 & 0 \\
\frac{1}{2} & -\frac{1}{2} & 0 & 0 & \frac{1}{2} & -\frac{1}{2} & 0 & 0 & -\frac{1}{2} & \frac{1}{2} & 0 & 0 & -\frac{1}{2} & \frac{1}{2} & 0 & 0 \\
1 & 0 & \frac{1}{2} & \frac{1}{2} & 1 & 0 & \frac{1}{2} & \frac{1}{2} & 0 & 1 & \frac{1}{2} & \frac{1}{2} & 0 & 1 & \frac{1}{2} & \frac{1}{2} \\
0 & -1 & -\frac{1}{2} & -\frac{1}{2} & 0 & -1 & -\frac{1}{2} & -\frac{1}{2} & -1 & 0 & -\frac{1}{2} & -\frac{1}{2} & -1 & 0 & -\frac{1}{2} & -\frac{1}{2} \\
\frac{1}{2} & -\frac{1}{2} & 0 & 0 & \frac{1}{2} & -\frac{1}{2} & 0 & 0 & -\frac{1}{2} & \frac{1}{2} & 0 & 0 & -\frac{1}{2} & \frac{1}{2} & 0 & 0 \\
\frac{1}{2} & -\frac{1}{2} & 0 & 0 & \frac{1}{2} & -\frac{1}{2} & 0 & 0 & -\frac{1}{2} & \frac{1}{2} & 0 & 0 & -\frac{1}{2} & \frac{1}{2} & 0 & 0
\end{bmatrix}_{22}$$

$$\left[\begin{array}{cccccccccccccccc}
0 & 0 & \frac{-\sqrt{3}}{2} & \frac{-1}{2\sqrt{3}} & 0 & 0 & \frac{-\sqrt{3}}{2} & \frac{-1}{2\sqrt{3}} & -\frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} & \frac{1}{2\sqrt{3}} & \frac{-1}{2\sqrt{3}} & -\frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} & \frac{1}{2\sqrt{3}} & \frac{-1}{2\sqrt{3}} \\
0 & 0 & \frac{-\sqrt{3}}{2} & \frac{-1}{2\sqrt{3}} & 0 & 0 & \frac{-\sqrt{3}}{2} & \frac{-1}{2\sqrt{3}} & -\frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} & \frac{1}{2\sqrt{3}} & \frac{-1}{2\sqrt{3}} & -\frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} & \frac{1}{2\sqrt{3}} & \frac{-1}{2\sqrt{3}} \\
\frac{\sqrt{3}}{2} & \frac{\sqrt{3}}{2} & 0 & \frac{1}{\sqrt{3}} & \frac{\sqrt{3}}{2} & \frac{\sqrt{3}}{2} & 0 & \frac{1}{\sqrt{3}} & \frac{1}{2\sqrt{3}} & \frac{1}{2\sqrt{3}} & \frac{2}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{2\sqrt{3}} & \frac{1}{2\sqrt{3}} & \frac{2}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\
\frac{1}{2\sqrt{3}} & \frac{1}{2\sqrt{3}} & -\frac{1}{\sqrt{3}} & 0 & \frac{1}{2\sqrt{3}} & \frac{1}{2\sqrt{3}} & -\frac{1}{\sqrt{3}} & 0 & \frac{-1}{2\sqrt{3}} & \frac{-1}{2\sqrt{3}} & \frac{1}{\sqrt{3}} & 0 & \frac{-1}{2\sqrt{3}} & \frac{-1}{2\sqrt{3}} & \frac{1}{\sqrt{3}} & 0 \\
0 & 0 & \frac{-\sqrt{3}}{2} & \frac{-1}{2\sqrt{3}} & 0 & 0 & \frac{-\sqrt{3}}{2} & \frac{-1}{2\sqrt{3}} & -\frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} & \frac{1}{2\sqrt{3}} & \frac{-1}{2\sqrt{3}} & -\frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} & \frac{1}{2\sqrt{3}} & \frac{-1}{2\sqrt{3}} \\
0 & 0 & \frac{-\sqrt{3}}{2} & \frac{-1}{2\sqrt{3}} & 0 & 0 & \frac{-\sqrt{3}}{2} & \frac{-1}{2\sqrt{3}} & -\frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} & \frac{1}{2\sqrt{3}} & \frac{-1}{2\sqrt{3}} & -\frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} & \frac{1}{2\sqrt{3}} & \frac{-1}{2\sqrt{3}} \\
\frac{\sqrt{3}}{2} & \frac{\sqrt{3}}{2} & 0 & \frac{1}{\sqrt{3}} & \frac{\sqrt{3}}{2} & \frac{\sqrt{3}}{2} & 0 & \frac{1}{\sqrt{3}} & \frac{1}{2\sqrt{3}} & \frac{1}{2\sqrt{3}} & \frac{2}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{2\sqrt{3}} & \frac{1}{2\sqrt{3}} & \frac{2}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\
\frac{1}{2\sqrt{3}} & \frac{1}{2\sqrt{3}} & -\frac{1}{\sqrt{3}} & 0 & \frac{1}{2\sqrt{3}} & \frac{1}{2\sqrt{3}} & -\frac{1}{\sqrt{3}} & 0 & \frac{-1}{2\sqrt{3}} & \frac{-1}{2\sqrt{3}} & \frac{1}{\sqrt{3}} & 0 & \frac{-1}{2\sqrt{3}} & \frac{-1}{2\sqrt{3}} & \frac{1}{\sqrt{3}} & 0 \\
\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{-1}{2\sqrt{3}} & \frac{1}{2\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{-1}{2\sqrt{3}} & \frac{1}{2\sqrt{3}} & 0 & 0 & \frac{\sqrt{3}}{2} & \frac{1}{2\sqrt{3}} & 0 & 0 & \frac{\sqrt{3}}{2} & \frac{1}{2\sqrt{3}} \\
\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{-1}{2\sqrt{3}} & \frac{1}{2\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{-1}{2\sqrt{3}} & \frac{1}{2\sqrt{3}} & 0 & 0 & \frac{\sqrt{3}}{2} & \frac{1}{2\sqrt{3}} & 0 & 0 & \frac{\sqrt{3}}{2} & \frac{1}{2\sqrt{3}} \\
\frac{-1}{2\sqrt{3}} & \frac{-1}{2\sqrt{3}} & \frac{-2}{\sqrt{3}} & -\frac{1}{\sqrt{3}} & \frac{-1}{2\sqrt{3}} & \frac{-1}{2\sqrt{3}} & \frac{-2}{\sqrt{3}} & -\frac{1}{\sqrt{3}} & \frac{-\sqrt{3}}{2} & \frac{-\sqrt{3}}{2} & 0 & -\frac{1}{\sqrt{3}} & \frac{-\sqrt{3}}{2} & \frac{-\sqrt{3}}{2} & 0 & -\frac{1}{\sqrt{3}} \\
\frac{1}{2\sqrt{3}} & \frac{1}{2\sqrt{3}} & -\frac{1}{\sqrt{3}} & 0 & \frac{1}{2\sqrt{3}} & \frac{1}{2\sqrt{3}} & -\frac{1}{\sqrt{3}} & 0 & \frac{-1}{2\sqrt{3}} & \frac{-1}{2\sqrt{3}} & \frac{1}{\sqrt{3}} & 0 & \frac{-1}{2\sqrt{3}} & \frac{-1}{2\sqrt{3}} & \frac{1}{\sqrt{3}} & 0 \\
\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{-1}{2\sqrt{3}} & \frac{1}{2\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{-1}{2\sqrt{3}} & \frac{1}{2\sqrt{3}} & 0 & 0 & \frac{\sqrt{3}}{2} & \frac{1}{2\sqrt{3}} & 0 & 0 & \frac{\sqrt{3}}{2} & \frac{1}{2\sqrt{3}} \\
\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{-1}{2\sqrt{3}} & \frac{1}{2\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{-1}{2\sqrt{3}} & \frac{1}{2\sqrt{3}} & 0 & 0 & \frac{\sqrt{3}}{2} & \frac{1}{2\sqrt{3}} & 0 & 0 & \frac{\sqrt{3}}{2} & \frac{1}{2\sqrt{3}} \\
\frac{-1}{2\sqrt{3}} & \frac{-1}{2\sqrt{3}} & \frac{-2}{\sqrt{3}} & -\frac{1}{\sqrt{3}} & \frac{-1}{2\sqrt{3}} & \frac{-1}{2\sqrt{3}} & \frac{-2}{\sqrt{3}} & -\frac{1}{\sqrt{3}} & \frac{-\sqrt{3}}{2} & \frac{-\sqrt{3}}{2} & 0 & -\frac{1}{\sqrt{3}} & \frac{-\sqrt{3}}{2} & \frac{-\sqrt{3}}{2} & 0 & -\frac{1}{\sqrt{3}} \\
\frac{1}{2\sqrt{3}} & \frac{1}{2\sqrt{3}} & -\frac{1}{\sqrt{3}} & 0 & \frac{1}{2\sqrt{3}} & \frac{1}{2\sqrt{3}} & -\frac{1}{\sqrt{3}} & 0 & \frac{-1}{2\sqrt{3}} & \frac{-1}{2\sqrt{3}} & \frac{1}{\sqrt{3}} & 0 & \frac{-1}{2\sqrt{3}} & \frac{-1}{2\sqrt{3}} & \frac{1}{\sqrt{3}} & 0
\end{array}\right]_{11}$$

$$\left[\begin{array}{cccccccccccccccc}
0 & 0 & \frac{-\sqrt{3}}{2} & \frac{-1}{2\sqrt{3}} & 0 & 0 & \frac{-\sqrt{3}}{2} & \frac{-1}{2\sqrt{3}} & -\frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} & \frac{1}{2\sqrt{3}} & \frac{-1}{2\sqrt{3}} & -\frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} & \frac{1}{2\sqrt{3}} & \frac{-1}{2\sqrt{3}} \\
0 & 0 & \frac{-\sqrt{3}}{2} & \frac{-1}{2\sqrt{3}} & 0 & 0 & \frac{-\sqrt{3}}{2} & \frac{-1}{2\sqrt{3}} & -\frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} & \frac{1}{2\sqrt{3}} & \frac{-1}{2\sqrt{3}} & -\frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} & \frac{1}{2\sqrt{3}} & \frac{-1}{2\sqrt{3}} \\
\frac{\sqrt{3}}{2} & \frac{\sqrt{3}}{2} & 0 & \frac{1}{\sqrt{3}} & \frac{\sqrt{3}}{2} & \frac{\sqrt{3}}{2} & 0 & \frac{1}{\sqrt{3}} & \frac{1}{2\sqrt{3}} & \frac{1}{2\sqrt{3}} & \frac{2}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{2\sqrt{3}} & \frac{1}{2\sqrt{3}} & \frac{2}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\
\frac{1}{2\sqrt{3}} & \frac{1}{2\sqrt{3}} & -\frac{1}{\sqrt{3}} & 0 & \frac{1}{2\sqrt{3}} & \frac{1}{2\sqrt{3}} & -\frac{1}{\sqrt{3}} & 0 & \frac{-1}{2\sqrt{3}} & \frac{-1}{2\sqrt{3}} & \frac{1}{\sqrt{3}} & 0 & \frac{-1}{2\sqrt{3}} & \frac{-1}{2\sqrt{3}} & \frac{1}{\sqrt{3}} & 0 \\
0 & 0 & \frac{-\sqrt{3}}{2} & \frac{-1}{2\sqrt{3}} & 0 & 0 & \frac{-\sqrt{3}}{2} & \frac{-1}{2\sqrt{3}} & -\frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} & \frac{1}{2\sqrt{3}} & \frac{-1}{2\sqrt{3}} & -\frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} & \frac{1}{2\sqrt{3}} & \frac{-1}{2\sqrt{3}} \\
0 & 0 & \frac{-\sqrt{3}}{2} & \frac{-1}{2\sqrt{3}} & 0 & 0 & \frac{-\sqrt{3}}{2} & \frac{-1}{2\sqrt{3}} & -\frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} & \frac{1}{2\sqrt{3}} & \frac{-1}{2\sqrt{3}} & -\frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} & \frac{1}{2\sqrt{3}} & \frac{-1}{2\sqrt{3}} \\
\frac{\sqrt{3}}{2} & \frac{\sqrt{3}}{2} & 0 & \frac{1}{\sqrt{3}} & \frac{\sqrt{3}}{2} & \frac{\sqrt{3}}{2} & 0 & \frac{1}{\sqrt{3}} & \frac{1}{2\sqrt{3}} & \frac{1}{2\sqrt{3}} & \frac{2}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{2\sqrt{3}} & \frac{1}{2\sqrt{3}} & \frac{2}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\
\frac{1}{2\sqrt{3}} & \frac{1}{2\sqrt{3}} & -\frac{1}{\sqrt{3}} & 0 & \frac{1}{2\sqrt{3}} & \frac{1}{2\sqrt{3}} & -\frac{1}{\sqrt{3}} & 0 & \frac{-1}{2\sqrt{3}} & \frac{-1}{2\sqrt{3}} & \frac{1}{\sqrt{3}} & 0 & \frac{-1}{2\sqrt{3}} & \frac{-1}{2\sqrt{3}} & \frac{1}{\sqrt{3}} & 0 \\
\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{-1}{2\sqrt{3}} & \frac{1}{2\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{-1}{2\sqrt{3}} & \frac{1}{2\sqrt{3}} & 0 & 0 & \frac{\sqrt{3}}{2} & \frac{1}{2\sqrt{3}} & 0 & 0 & \frac{\sqrt{3}}{2} & \frac{1}{2\sqrt{3}} \\
\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{-1}{2\sqrt{3}} & \frac{1}{2\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{-1}{2\sqrt{3}} & \frac{1}{2\sqrt{3}} & 0 & 0 & \frac{\sqrt{3}}{2} & \frac{1}{2\sqrt{3}} & 0 & 0 & \frac{\sqrt{3}}{2} & \frac{1}{2\sqrt{3}} \\
\frac{-1}{2\sqrt{3}} & \frac{-1}{2\sqrt{3}} & \frac{-2}{\sqrt{3}} & -\frac{1}{\sqrt{3}} & \frac{-1}{2\sqrt{3}} & \frac{-1}{2\sqrt{3}} & \frac{-2}{\sqrt{3}} & -\frac{1}{\sqrt{3}} & \frac{-\sqrt{3}}{2} & \frac{-\sqrt{3}}{2} & 0 & -\frac{1}{\sqrt{3}} & \frac{-\sqrt{3}}{2} & \frac{-\sqrt{3}}{2} & 0 & -\frac{1}{\sqrt{3}} \\
\frac{1}{2\sqrt{3}} & \frac{1}{2\sqrt{3}} & -\frac{1}{\sqrt{3}} & 0 & \frac{1}{2\sqrt{3}} & \frac{1}{2\sqrt{3}} & -\frac{1}{\sqrt{3}} & 0 & \frac{-1}{2\sqrt{3}} & \frac{-1}{2\sqrt{3}} & \frac{1}{\sqrt{3}} & 0 & \frac{-1}{2\sqrt{3}} & \frac{-1}{2\sqrt{3}} & \frac{1}{\sqrt{3}} & 0 \\
\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{-1}{2\sqrt{3}} & \frac{1}{2\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{-1}{2\sqrt{3}} & \frac{1}{2\sqrt{3}} & 0 & 0 & \frac{\sqrt{3}}{2} & \frac{1}{2\sqrt{3}} & 0 & 0 & \frac{\sqrt{3}}{2} & \frac{1}{2\sqrt{3}} \\
\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{-1}{2\sqrt{3}} & \frac{1}{2\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{-1}{2\sqrt{3}} & \frac{1}{2\sqrt{3}} & 0 & 0 & \frac{\sqrt{3}}{2} & \frac{1}{2\sqrt{3}} & 0 & 0 & \frac{\sqrt{3}}{2} & \frac{1}{2\sqrt{3}} \\
\frac{-1}{2\sqrt{3}} & \frac{-1}{2\sqrt{3}} & \frac{-2}{\sqrt{3}} & -\frac{1}{\sqrt{3}} & \frac{-1}{2\sqrt{3}} & \frac{-1}{2\sqrt{3}} & \frac{-2}{\sqrt{3}} & -\frac{1}{\sqrt{3}} & \frac{-\sqrt{3}}{2} & \frac{-\sqrt{3}}{2} & 0 & -\frac{1}{\sqrt{3}} & \frac{-\sqrt{3}}{2} & \frac{-\sqrt{3}}{2} & 0 & -\frac{1}{\sqrt{3}} \\
\frac{1}{2\sqrt{3}} & \frac{1}{2\sqrt{3}} & -\frac{1}{\sqrt{3}} & 0 & \frac{1}{2\sqrt{3}} & \frac{1}{2\sqrt{3}} & -\frac{1}{\sqrt{3}} & 0 & \frac{-1}{2\sqrt{3}} & \frac{-1}{2\sqrt{3}} & \frac{1}{\sqrt{3}} & 0 & \frac{-1}{2\sqrt{3}} & \frac{-1}{2\sqrt{3}} & \frac{1}{\sqrt{3}} & 0
\end{array}\right]_{21}$$

$$\begin{bmatrix}
0 & 0 & \frac{-\sqrt{3}}{2} & \frac{-1}{2\sqrt{3}} & 0 & 0 & \frac{-\sqrt{3}}{2} & \frac{-1}{2\sqrt{3}} & -\frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} & \frac{1}{2\sqrt{3}} & \frac{-1}{2\sqrt{3}} & -\frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} & \frac{1}{2\sqrt{3}} & \frac{-1}{2\sqrt{3}} \\
0 & 0 & \frac{-\sqrt{3}}{2} & \frac{-1}{2\sqrt{3}} & 0 & 0 & \frac{-\sqrt{3}}{2} & \frac{-1}{2\sqrt{3}} & -\frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} & \frac{1}{2\sqrt{3}} & \frac{-1}{2\sqrt{3}} & -\frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} & \frac{1}{2\sqrt{3}} & \frac{-1}{2\sqrt{3}} \\
\frac{\sqrt{3}}{2} & \frac{\sqrt{3}}{2} & 0 & \frac{1}{\sqrt{3}} & \frac{\sqrt{3}}{2} & \frac{\sqrt{3}}{2} & 0 & \frac{1}{\sqrt{3}} & \frac{1}{2\sqrt{3}} & \frac{1}{2\sqrt{3}} & \frac{2}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{2\sqrt{3}} & \frac{1}{2\sqrt{3}} & \frac{2}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\
\frac{1}{2\sqrt{3}} & \frac{1}{2\sqrt{3}} & -\frac{1}{\sqrt{3}} & 0 & \frac{1}{2\sqrt{3}} & \frac{1}{2\sqrt{3}} & -\frac{1}{\sqrt{3}} & 0 & \frac{-1}{2\sqrt{3}} & \frac{-1}{2\sqrt{3}} & \frac{1}{\sqrt{3}} & 0 & \frac{-1}{2\sqrt{3}} & \frac{-1}{2\sqrt{3}} & \frac{1}{\sqrt{3}} & 0 \\
0 & 0 & \frac{-\sqrt{3}}{2} & \frac{-1}{2\sqrt{3}} & 0 & 0 & \frac{-\sqrt{3}}{2} & \frac{-1}{2\sqrt{3}} & -\frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} & \frac{1}{2\sqrt{3}} & \frac{-1}{2\sqrt{3}} & -\frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} & \frac{1}{2\sqrt{3}} & \frac{-1}{2\sqrt{3}} \\
0 & 0 & \frac{-\sqrt{3}}{2} & \frac{-1}{2\sqrt{3}} & 0 & 0 & \frac{-\sqrt{3}}{2} & \frac{-1}{2\sqrt{3}} & -\frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} & \frac{1}{2\sqrt{3}} & \frac{-1}{2\sqrt{3}} & -\frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} & \frac{1}{2\sqrt{3}} & \frac{-1}{2\sqrt{3}} \\
\frac{\sqrt{3}}{2} & \frac{\sqrt{3}}{2} & 0 & \frac{1}{\sqrt{3}} & \frac{\sqrt{3}}{2} & \frac{\sqrt{3}}{2} & 0 & \frac{1}{\sqrt{3}} & \frac{1}{2\sqrt{3}} & \frac{1}{2\sqrt{3}} & \frac{2}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{2\sqrt{3}} & \frac{1}{2\sqrt{3}} & \frac{2}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\
\frac{1}{2\sqrt{3}} & \frac{1}{2\sqrt{3}} & -\frac{1}{\sqrt{3}} & 0 & \frac{1}{2\sqrt{3}} & \frac{1}{2\sqrt{3}} & -\frac{1}{\sqrt{3}} & 0 & \frac{-1}{2\sqrt{3}} & \frac{-1}{2\sqrt{3}} & \frac{1}{\sqrt{3}} & 0 & \frac{-1}{2\sqrt{3}} & \frac{-1}{2\sqrt{3}} & \frac{1}{\sqrt{3}} & 0 \\
\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{-1}{2\sqrt{3}} & \frac{1}{2\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{-1}{2\sqrt{3}} & \frac{1}{2\sqrt{3}} & 0 & 0 & \frac{\sqrt{3}}{2} & \frac{1}{2\sqrt{3}} & 0 & 0 & \frac{\sqrt{3}}{2} & \frac{1}{2\sqrt{3}} \\
\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{-1}{2\sqrt{3}} & \frac{1}{2\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{-1}{2\sqrt{3}} & \frac{1}{2\sqrt{3}} & 0 & 0 & \frac{\sqrt{3}}{2} & \frac{1}{2\sqrt{3}} & 0 & 0 & \frac{\sqrt{3}}{2} & \frac{1}{2\sqrt{3}} \\
\frac{-1}{2\sqrt{3}} & \frac{-1}{2\sqrt{3}} & \frac{-2}{\sqrt{3}} & -\frac{1}{\sqrt{3}} & \frac{-1}{2\sqrt{3}} & \frac{-1}{2\sqrt{3}} & \frac{-2}{\sqrt{3}} & -\frac{1}{\sqrt{3}} & \frac{-\sqrt{3}}{2} & \frac{-\sqrt{3}}{2} & 0 & -\frac{1}{\sqrt{3}} & \frac{-\sqrt{3}}{2} & \frac{-\sqrt{3}}{2} & 0 & -\frac{1}{\sqrt{3}} \\
\frac{1}{2\sqrt{3}} & \frac{1}{2\sqrt{3}} & -\frac{1}{\sqrt{3}} & 0 & \frac{1}{2\sqrt{3}} & \frac{1}{2\sqrt{3}} & -\frac{1}{\sqrt{3}} & 0 & \frac{-1}{2\sqrt{3}} & \frac{-1}{2\sqrt{3}} & \frac{1}{\sqrt{3}} & 0 & \frac{-1}{2\sqrt{3}} & \frac{-1}{2\sqrt{3}} & \frac{1}{\sqrt{3}} & 0 \\
\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{-1}{2\sqrt{3}} & \frac{1}{2\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{-1}{2\sqrt{3}} & \frac{1}{2\sqrt{3}} & 0 & 0 & \frac{\sqrt{3}}{2} & \frac{1}{2\sqrt{3}} & 0 & 0 & \frac{\sqrt{3}}{2} & \frac{1}{2\sqrt{3}} \\
\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{-1}{2\sqrt{3}} & \frac{1}{2\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{-1}{2\sqrt{3}} & \frac{1}{2\sqrt{3}} & 0 & 0 & \frac{\sqrt{3}}{2} & \frac{1}{2\sqrt{3}} & 0 & 0 & \frac{\sqrt{3}}{2} & \frac{1}{2\sqrt{3}} \\
\frac{-1}{2\sqrt{3}} & \frac{-1}{2\sqrt{3}} & \frac{-2}{\sqrt{3}} & -\frac{1}{\sqrt{3}} & \frac{-1}{2\sqrt{3}} & \frac{-1}{2\sqrt{3}} & \frac{-2}{\sqrt{3}} & -\frac{1}{\sqrt{3}} & \frac{-\sqrt{3}}{2} & \frac{-\sqrt{3}}{2} & 0 & -\frac{1}{\sqrt{3}} & \frac{-\sqrt{3}}{2} & \frac{-\sqrt{3}}{2} & 0 & -\frac{1}{\sqrt{3}} \\
\frac{1}{2\sqrt{3}} & \frac{1}{2\sqrt{3}} & -\frac{1}{\sqrt{3}} & 0 & \frac{1}{2\sqrt{3}} & \frac{1}{2\sqrt{3}} & -\frac{1}{\sqrt{3}} & 0 & \frac{-1}{2\sqrt{3}} & \frac{-1}{2\sqrt{3}} & \frac{1}{\sqrt{3}} & 0 & \frac{-1}{2\sqrt{3}} & \frac{-1}{2\sqrt{3}} & \frac{1}{\sqrt{3}} & 0
\end{bmatrix}_{12}$$

$$\begin{bmatrix}
0 & 0 & \frac{-\sqrt{3}}{2} & \frac{-1}{2\sqrt{3}} & 0 & 0 & \frac{-\sqrt{3}}{2} & \frac{-1}{2\sqrt{3}} & -\frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} & \frac{1}{2\sqrt{3}} & \frac{-1}{2\sqrt{3}} & -\frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} & \frac{1}{2\sqrt{3}} & \frac{-1}{2\sqrt{3}} \\
0 & 0 & \frac{-\sqrt{3}}{2} & \frac{-1}{2\sqrt{3}} & 0 & 0 & \frac{-\sqrt{3}}{2} & \frac{-1}{2\sqrt{3}} & -\frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} & \frac{1}{2\sqrt{3}} & \frac{-1}{2\sqrt{3}} & -\frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} & \frac{1}{2\sqrt{3}} & \frac{-1}{2\sqrt{3}} \\
\frac{\sqrt{3}}{2} & \frac{\sqrt{3}}{2} & 0 & \frac{1}{\sqrt{3}} & \frac{\sqrt{3}}{2} & \frac{\sqrt{3}}{2} & 0 & \frac{1}{\sqrt{3}} & \frac{1}{2\sqrt{3}} & \frac{1}{2\sqrt{3}} & \frac{2}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{2\sqrt{3}} & \frac{1}{2\sqrt{3}} & \frac{2}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\
\frac{1}{2\sqrt{3}} & \frac{1}{2\sqrt{3}} & -\frac{1}{\sqrt{3}} & 0 & \frac{1}{2\sqrt{3}} & \frac{1}{2\sqrt{3}} & -\frac{1}{\sqrt{3}} & 0 & \frac{-1}{2\sqrt{3}} & \frac{-1}{2\sqrt{3}} & \frac{1}{\sqrt{3}} & 0 & \frac{-1}{2\sqrt{3}} & \frac{-1}{2\sqrt{3}} & \frac{1}{\sqrt{3}} & 0 \\
0 & 0 & \frac{-\sqrt{3}}{2} & \frac{-1}{2\sqrt{3}} & 0 & 0 & \frac{-\sqrt{3}}{2} & \frac{-1}{2\sqrt{3}} & -\frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} & \frac{1}{2\sqrt{3}} & \frac{-1}{2\sqrt{3}} & -\frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} & \frac{1}{2\sqrt{3}} & \frac{-1}{2\sqrt{3}} \\
0 & 0 & \frac{-\sqrt{3}}{2} & \frac{-1}{2\sqrt{3}} & 0 & 0 & \frac{-\sqrt{3}}{2} & \frac{-1}{2\sqrt{3}} & -\frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} & \frac{1}{2\sqrt{3}} & \frac{-1}{2\sqrt{3}} & -\frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} & \frac{1}{2\sqrt{3}} & \frac{-1}{2\sqrt{3}} \\
\frac{\sqrt{3}}{2} & \frac{\sqrt{3}}{2} & 0 & \frac{1}{\sqrt{3}} & \frac{\sqrt{3}}{2} & \frac{\sqrt{3}}{2} & 0 & \frac{1}{\sqrt{3}} & \frac{1}{2\sqrt{3}} & \frac{1}{2\sqrt{3}} & \frac{2}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{2\sqrt{3}} & \frac{1}{2\sqrt{3}} & \frac{2}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\
\frac{1}{2\sqrt{3}} & \frac{1}{2\sqrt{3}} & -\frac{1}{\sqrt{3}} & 0 & \frac{1}{2\sqrt{3}} & \frac{1}{2\sqrt{3}} & -\frac{1}{\sqrt{3}} & 0 & \frac{-1}{2\sqrt{3}} & \frac{-1}{2\sqrt{3}} & \frac{1}{\sqrt{3}} & 0 & \frac{-1}{2\sqrt{3}} & \frac{-1}{2\sqrt{3}} & \frac{1}{\sqrt{3}} & 0 \\
\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{-1}{2\sqrt{3}} & \frac{1}{2\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{-1}{2\sqrt{3}} & \frac{1}{2\sqrt{3}} & 0 & 0 & \frac{\sqrt{3}}{2} & \frac{1}{2\sqrt{3}} & 0 & 0 & \frac{\sqrt{3}}{2} & \frac{1}{2\sqrt{3}} \\
\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{-1}{2\sqrt{3}} & \frac{1}{2\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{-1}{2\sqrt{3}} & \frac{1}{2\sqrt{3}} & 0 & 0 & \frac{\sqrt{3}}{2} & \frac{1}{2\sqrt{3}} & 0 & 0 & \frac{\sqrt{3}}{2} & \frac{1}{2\sqrt{3}} \\
\frac{-1}{2\sqrt{3}} & \frac{-1}{2\sqrt{3}} & \frac{-2}{\sqrt{3}} & -\frac{1}{\sqrt{3}} & \frac{-1}{2\sqrt{3}} & \frac{-1}{2\sqrt{3}} & \frac{-2}{\sqrt{3}} & -\frac{1}{\sqrt{3}} & \frac{-\sqrt{3}}{2} & \frac{-\sqrt{3}}{2} & 0 & -\frac{1}{\sqrt{3}} & \frac{-\sqrt{3}}{2} & \frac{-\sqrt{3}}{2} & 0 & -\frac{1}{\sqrt{3}} \\
\frac{1}{2\sqrt{3}} & \frac{1}{2\sqrt{3}} & -\frac{1}{\sqrt{3}} & 0 & \frac{1}{2\sqrt{3}} & \frac{1}{2\sqrt{3}} & -\frac{1}{\sqrt{3}} & 0 & \frac{-1}{2\sqrt{3}} & \frac{-1}{2\sqrt{3}} & \frac{1}{\sqrt{3}} & 0 & \frac{-1}{2\sqrt{3}} & \frac{-1}{2\sqrt{3}} & \frac{1}{\sqrt{3}} & 0 \\
\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{-1}{2\sqrt{3}} & \frac{1}{2\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{-1}{2\sqrt{3}} & \frac{1}{2\sqrt{3}} & 0 & 0 & \frac{\sqrt{3}}{2} & \frac{1}{2\sqrt{3}} & 0 & 0 & \frac{\sqrt{3}}{2} & \frac{1}{2\sqrt{3}} \\
\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{-1}{2\sqrt{3}} & \frac{1}{2\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{-1}{2\sqrt{3}} & \frac{1}{2\sqrt{3}} & 0 & 0 & \frac{\sqrt{3}}{2} & \frac{1}{2\sqrt{3}} & 0 & 0 & \frac{\sqrt{3}}{2} & \frac{1}{2\sqrt{3}} \\
\frac{-1}{2\sqrt{3}} & \frac{-1}{2\sqrt{3}} & \frac{-2}{\sqrt{3}} & -\frac{1}{\sqrt{3}} & \frac{-1}{2\sqrt{3}} & \frac{-1}{2\sqrt{3}} & \frac{-2}{\sqrt{3}} & -\frac{1}{\sqrt{3}} & \frac{-\sqrt{3}}{2} & \frac{-\sqrt{3}}{2} & 0 & -\frac{1}{\sqrt{3}} & \frac{-\sqrt{3}}{2} & \frac{-\sqrt{3}}{2} & 0 & -\frac{1}{\sqrt{3}} \\
\frac{1}{2\sqrt{3}} & \frac{1}{2\sqrt{3}} & -\frac{1}{\sqrt{3}} & 0 & \frac{1}{2\sqrt{3}} & \frac{1}{2\sqrt{3}} & -\frac{1}{\sqrt{3}} & 0 & \frac{-1}{2\sqrt{3}} & \frac{-1}{2\sqrt{3}} & \frac{1}{\sqrt{3}} & 0 & \frac{-1}{2\sqrt{3}} & \frac{-1}{2\sqrt{3}} & \frac{1}{\sqrt{3}} & 0
\end{bmatrix}_{22}$$

$$\begin{bmatrix}
0 & 0 & 0 & -\sqrt{\frac{2}{3}} & 0 & 0 & 0 & -\sqrt{\frac{2}{3}} & -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\
0 & 0 & 0 & -\sqrt{\frac{2}{3}} & 0 & 0 & 0 & -\sqrt{\frac{2}{3}} & -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\
0 & 0 & 0 & -\sqrt{\frac{2}{3}} & 0 & 0 & 0 & -\sqrt{\frac{2}{3}} & -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\
\sqrt{\frac{2}{3}} & \sqrt{\frac{2}{3}} & \sqrt{\frac{2}{3}} & 0 & \sqrt{\frac{2}{3}} & \sqrt{\frac{2}{3}} & \sqrt{\frac{2}{3}} & 0 & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \sqrt{\frac{3}{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \sqrt{\frac{3}{2}} \\
0 & 0 & 0 & -\sqrt{\frac{2}{3}} & 0 & 0 & 0 & -\sqrt{\frac{2}{3}} & -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\
0 & 0 & 0 & -\sqrt{\frac{2}{3}} & 0 & 0 & 0 & -\sqrt{\frac{2}{3}} & -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\
0 & 0 & 0 & -\sqrt{\frac{2}{3}} & 0 & 0 & 0 & -\sqrt{\frac{2}{3}} & -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\
\sqrt{\frac{2}{3}} & \sqrt{\frac{2}{3}} & \sqrt{\frac{2}{3}} & 0 & \sqrt{\frac{2}{3}} & \sqrt{\frac{2}{3}} & \sqrt{\frac{2}{3}} & 0 & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \sqrt{\frac{3}{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \sqrt{\frac{3}{2}} \\
\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & 0 & 0 & 0 & \sqrt{\frac{2}{3}} & 0 & 0 & 0 & \sqrt{\frac{2}{3}} \\
\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & 0 & 0 & 0 & \sqrt{\frac{2}{3}} & 0 & 0 & 0 & \sqrt{\frac{2}{3}} \\
\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & 0 & 0 & 0 & \sqrt{\frac{2}{3}} & 0 & 0 & 0 & \sqrt{\frac{2}{3}} \\
-\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & -\sqrt{\frac{3}{2}} & -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & -\sqrt{\frac{3}{2}} & -\sqrt{\frac{2}{3}} & -\sqrt{\frac{2}{3}} & -\sqrt{\frac{2}{3}} & 0 & -\sqrt{\frac{2}{3}} & -\sqrt{\frac{2}{3}} & -\sqrt{\frac{2}{3}} & 0 \\
\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & 0 & 0 & 0 & \sqrt{\frac{2}{3}} & 0 & 0 & 0 & \sqrt{\frac{2}{3}} \\
\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & 0 & 0 & 0 & \sqrt{\frac{2}{3}} & 0 & 0 & 0 & \sqrt{\frac{2}{3}} \\
\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & 0 & 0 & 0 & \sqrt{\frac{2}{3}} & 0 & 0 & 0 & \sqrt{\frac{2}{3}} \\
-\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & -\sqrt{\frac{3}{2}} & -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & -\sqrt{\frac{3}{2}} & -\sqrt{\frac{2}{3}} & -\sqrt{\frac{2}{3}} & -\sqrt{\frac{2}{3}} & 0 & -\sqrt{\frac{2}{3}} & -\sqrt{\frac{2}{3}} & -\sqrt{\frac{2}{3}} & 0
\end{bmatrix}_{11}$$

$$\begin{bmatrix}
0 & 0 & 0 & -\sqrt{\frac{2}{3}} & 0 & 0 & 0 & -\sqrt{\frac{2}{3}} & -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\
0 & 0 & 0 & -\sqrt{\frac{2}{3}} & 0 & 0 & 0 & -\sqrt{\frac{2}{3}} & -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\
0 & 0 & 0 & -\sqrt{\frac{2}{3}} & 0 & 0 & 0 & -\sqrt{\frac{2}{3}} & -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\
\sqrt{\frac{2}{3}} & \sqrt{\frac{2}{3}} & \sqrt{\frac{2}{3}} & 0 & \sqrt{\frac{2}{3}} & \sqrt{\frac{2}{3}} & \sqrt{\frac{2}{3}} & 0 & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \sqrt{\frac{3}{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \sqrt{\frac{3}{2}} \\
0 & 0 & 0 & -\sqrt{\frac{2}{3}} & 0 & 0 & 0 & -\sqrt{\frac{2}{3}} & -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\
0 & 0 & 0 & -\sqrt{\frac{2}{3}} & 0 & 0 & 0 & -\sqrt{\frac{2}{3}} & -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\
0 & 0 & 0 & -\sqrt{\frac{2}{3}} & 0 & 0 & 0 & -\sqrt{\frac{2}{3}} & -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\
\sqrt{\frac{2}{3}} & \sqrt{\frac{2}{3}} & \sqrt{\frac{2}{3}} & 0 & \sqrt{\frac{2}{3}} & \sqrt{\frac{2}{3}} & \sqrt{\frac{2}{3}} & 0 & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \sqrt{\frac{3}{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \sqrt{\frac{3}{2}} \\
\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & 0 & 0 & 0 & \sqrt{\frac{2}{3}} & 0 & 0 & 0 & \sqrt{\frac{2}{3}} \\
\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & 0 & 0 & 0 & \sqrt{\frac{2}{3}} & 0 & 0 & 0 & \sqrt{\frac{2}{3}} \\
\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & 0 & 0 & 0 & \sqrt{\frac{2}{3}} & 0 & 0 & 0 & \sqrt{\frac{2}{3}} \\
-\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & -\sqrt{\frac{3}{2}} & -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & -\sqrt{\frac{3}{2}} & -\sqrt{\frac{2}{3}} & -\sqrt{\frac{2}{3}} & -\sqrt{\frac{2}{3}} & 0 & -\sqrt{\frac{2}{3}} & -\sqrt{\frac{2}{3}} & -\sqrt{\frac{2}{3}} & 0 \\
\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & 0 & 0 & 0 & \sqrt{\frac{2}{3}} & 0 & 0 & 0 & \sqrt{\frac{2}{3}} \\
\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & 0 & 0 & 0 & \sqrt{\frac{2}{3}} & 0 & 0 & 0 & \sqrt{\frac{2}{3}} \\
\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & 0 & 0 & 0 & \sqrt{\frac{2}{3}} & 0 & 0 & 0 & \sqrt{\frac{2}{3}} \\
-\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & -\sqrt{\frac{3}{2}} & -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & -\sqrt{\frac{3}{2}} & -\sqrt{\frac{2}{3}} & -\sqrt{\frac{2}{3}} & -\sqrt{\frac{2}{3}} & 0 & -\sqrt{\frac{2}{3}} & -\sqrt{\frac{2}{3}} & -\sqrt{\frac{2}{3}} & 0
\end{bmatrix}_{21}$$

$$\begin{bmatrix}
0 & 0 & 0 & -\sqrt{\frac{2}{3}} & 0 & 0 & 0 & -\sqrt{\frac{2}{3}} & -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\
0 & 0 & 0 & -\sqrt{\frac{2}{3}} & 0 & 0 & 0 & -\sqrt{\frac{2}{3}} & -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\
0 & 0 & 0 & -\sqrt{\frac{2}{3}} & 0 & 0 & 0 & -\sqrt{\frac{2}{3}} & -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\
\sqrt{\frac{2}{3}} & \sqrt{\frac{2}{3}} & \sqrt{\frac{2}{3}} & 0 & \sqrt{\frac{2}{3}} & \sqrt{\frac{2}{3}} & \sqrt{\frac{2}{3}} & 0 & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \sqrt{\frac{3}{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \sqrt{\frac{3}{2}} \\
0 & 0 & 0 & -\sqrt{\frac{2}{3}} & 0 & 0 & 0 & -\sqrt{\frac{2}{3}} & -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\
0 & 0 & 0 & -\sqrt{\frac{2}{3}} & 0 & 0 & 0 & -\sqrt{\frac{2}{3}} & -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\
0 & 0 & 0 & -\sqrt{\frac{2}{3}} & 0 & 0 & 0 & -\sqrt{\frac{2}{3}} & -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\
\sqrt{\frac{2}{3}} & \sqrt{\frac{2}{3}} & \sqrt{\frac{2}{3}} & 0 & \sqrt{\frac{2}{3}} & \sqrt{\frac{2}{3}} & \sqrt{\frac{2}{3}} & 0 & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \sqrt{\frac{3}{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \sqrt{\frac{3}{2}} \\
\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & 0 & 0 & 0 & \sqrt{\frac{2}{3}} & 0 & 0 & 0 & \sqrt{\frac{2}{3}} \\
\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & 0 & 0 & 0 & \sqrt{\frac{2}{3}} & 0 & 0 & 0 & \sqrt{\frac{2}{3}} \\
\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & 0 & 0 & 0 & \sqrt{\frac{2}{3}} & 0 & 0 & 0 & \sqrt{\frac{2}{3}} \\
-\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & -\sqrt{\frac{3}{2}} & -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & -\sqrt{\frac{3}{2}} & -\sqrt{\frac{2}{3}} & -\sqrt{\frac{2}{3}} & -\sqrt{\frac{2}{3}} & 0 & -\sqrt{\frac{2}{3}} & -\sqrt{\frac{2}{3}} & -\sqrt{\frac{2}{3}} & 0 \\
\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & 0 & 0 & 0 & \sqrt{\frac{2}{3}} & 0 & 0 & 0 & \sqrt{\frac{2}{3}} \\
\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & 0 & 0 & 0 & \sqrt{\frac{2}{3}} & 0 & 0 & 0 & \sqrt{\frac{2}{3}} \\
\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & 0 & 0 & 0 & \sqrt{\frac{2}{3}} & 0 & 0 & 0 & \sqrt{\frac{2}{3}} \\
-\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & -\sqrt{\frac{3}{2}} & -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & -\sqrt{\frac{3}{2}} & -\sqrt{\frac{2}{3}} & -\sqrt{\frac{2}{3}} & -\sqrt{\frac{2}{3}} & 0 & -\sqrt{\frac{2}{3}} & -\sqrt{\frac{2}{3}} & -\sqrt{\frac{2}{3}} & 0
\end{bmatrix}_{12}$$

$$\begin{bmatrix}
0 & 0 & 0 & -\sqrt{\frac{2}{3}} & 0 & 0 & 0 & -\sqrt{\frac{2}{3}} & -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\
0 & 0 & 0 & -\sqrt{\frac{2}{3}} & 0 & 0 & 0 & -\sqrt{\frac{2}{3}} & -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\
0 & 0 & 0 & -\sqrt{\frac{2}{3}} & 0 & 0 & 0 & -\sqrt{\frac{2}{3}} & -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\
\sqrt{\frac{2}{3}} & \sqrt{\frac{2}{3}} & \sqrt{\frac{2}{3}} & 0 & \sqrt{\frac{2}{3}} & \sqrt{\frac{2}{3}} & \sqrt{\frac{2}{3}} & 0 & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \sqrt{\frac{3}{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \sqrt{\frac{3}{2}} \\
0 & 0 & 0 & -\sqrt{\frac{2}{3}} & 0 & 0 & 0 & -\sqrt{\frac{2}{3}} & -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\
0 & 0 & 0 & -\sqrt{\frac{2}{3}} & 0 & 0 & 0 & -\sqrt{\frac{2}{3}} & -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\
0 & 0 & 0 & -\sqrt{\frac{2}{3}} & 0 & 0 & 0 & -\sqrt{\frac{2}{3}} & -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\
\sqrt{\frac{2}{3}} & \sqrt{\frac{2}{3}} & \sqrt{\frac{2}{3}} & 0 & \sqrt{\frac{2}{3}} & \sqrt{\frac{2}{3}} & \sqrt{\frac{2}{3}} & 0 & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \sqrt{\frac{3}{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \sqrt{\frac{3}{2}} \\
\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & 0 & 0 & 0 & \sqrt{\frac{2}{3}} & 0 & 0 & 0 & \sqrt{\frac{2}{3}} \\
\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & 0 & 0 & 0 & \sqrt{\frac{2}{3}} & 0 & 0 & 0 & \sqrt{\frac{2}{3}} \\
\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & 0 & 0 & 0 & \sqrt{\frac{2}{3}} & 0 & 0 & 0 & \sqrt{\frac{2}{3}} \\
-\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & -\sqrt{\frac{3}{2}} & -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & -\sqrt{\frac{3}{2}} & -\sqrt{\frac{2}{3}} & -\sqrt{\frac{2}{3}} & -\sqrt{\frac{2}{3}} & 0 & -\sqrt{\frac{2}{3}} & -\sqrt{\frac{2}{3}} & -\sqrt{\frac{2}{3}} & 0 \\
\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & 0 & 0 & 0 & \sqrt{\frac{2}{3}} & 0 & 0 & 0 & \sqrt{\frac{2}{3}} \\
\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & 0 & 0 & 0 & \sqrt{\frac{2}{3}} & 0 & 0 & 0 & \sqrt{\frac{2}{3}} \\
\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & 0 & 0 & 0 & \sqrt{\frac{2}{3}} & 0 & 0 & 0 & \sqrt{\frac{2}{3}} \\
-\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & -\sqrt{\frac{3}{2}} & -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & -\sqrt{\frac{3}{2}} & -\sqrt{\frac{2}{3}} & -\sqrt{\frac{2}{3}} & -\sqrt{\frac{2}{3}} & 0 & -\sqrt{\frac{2}{3}} & -\sqrt{\frac{2}{3}} & -\sqrt{\frac{2}{3}} & 0
\end{bmatrix}_{22}$$

$$\begin{bmatrix}
0 & 0 & 0 & -\frac{2}{3} & -1 & -1 & -1 & -\frac{5}{3} & -\frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} & \frac{1}{3} & -\frac{4}{3} & -\frac{4}{3} & -\frac{4}{3} & -\frac{2}{3} \\
0 & 0 & 0 & -\frac{2}{3} & -1 & -1 & -1 & -\frac{5}{3} & -\frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} & \frac{1}{3} & -\frac{4}{3} & -\frac{4}{3} & -\frac{4}{3} & -\frac{2}{3} \\
0 & 0 & 0 & -\frac{2}{3} & -1 & -1 & -1 & -\frac{5}{3} & -\frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} & \frac{1}{3} & -\frac{4}{3} & -\frac{4}{3} & -\frac{4}{3} & -\frac{2}{3} \\
\frac{2}{3} & \frac{2}{3} & \frac{2}{3} & 0 & -\frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} & -1 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 1 & -\frac{2}{3} & -\frac{2}{3} & -\frac{2}{3} & 0 \\
1 & 1 & 1 & \frac{1}{3} & 0 & 0 & 0 & -\frac{2}{3} & \frac{2}{3} & \frac{2}{3} & \frac{2}{3} & \frac{4}{3} & -\frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} & \frac{1}{3} \\
1 & 1 & 1 & \frac{1}{3} & 0 & 0 & 0 & -\frac{2}{3} & \frac{2}{3} & \frac{2}{3} & \frac{2}{3} & \frac{4}{3} & -\frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} & \frac{1}{3} \\
1 & 1 & 1 & \frac{1}{3} & 0 & 0 & 0 & -\frac{2}{3} & \frac{2}{3} & \frac{2}{3} & \frac{2}{3} & \frac{4}{3} & -\frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} & \frac{1}{3} \\
\frac{5}{3} & \frac{5}{3} & \frac{5}{3} & 1 & \frac{2}{3} & \frac{2}{3} & \frac{2}{3} & 0 & \frac{4}{3} & \frac{4}{3} & \frac{4}{3} & 2 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 1 \\
\frac{1}{3} & \frac{1}{3} & \frac{1}{3} & -\frac{1}{3} & -\frac{2}{3} & -\frac{2}{3} & -\frac{2}{3} & -\frac{4}{3} & 0 & 0 & 0 & \frac{2}{3} & -1 & -1 & -1 & -\frac{1}{3} \\
\frac{1}{3} & \frac{1}{3} & \frac{1}{3} & -\frac{1}{3} & -\frac{2}{3} & -\frac{2}{3} & -\frac{2}{3} & -\frac{4}{3} & 0 & 0 & 0 & \frac{2}{3} & -1 & -1 & -1 & -\frac{1}{3} \\
\frac{1}{3} & \frac{1}{3} & \frac{1}{3} & -\frac{1}{3} & -\frac{2}{3} & -\frac{2}{3} & -\frac{2}{3} & -\frac{4}{3} & 0 & 0 & 0 & \frac{2}{3} & -1 & -1 & -1 & -\frac{1}{3} \\
-\frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} & -1 & -\frac{4}{3} & -\frac{4}{3} & -\frac{4}{3} & -2 & -\frac{2}{3} & -\frac{2}{3} & -\frac{2}{3} & 0 & -\frac{5}{3} & -\frac{5}{3} & -\frac{5}{3} & -1 \\
\frac{4}{3} & \frac{4}{3} & \frac{4}{3} & \frac{2}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & -\frac{1}{3} & 1 & 1 & 1 & \frac{5}{3} & 0 & 0 & 0 & \frac{2}{3} \\
\frac{4}{3} & \frac{4}{3} & \frac{4}{3} & \frac{2}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & -\frac{1}{3} & 1 & 1 & 1 & \frac{5}{3} & 0 & 0 & 0 & \frac{2}{3} \\
\frac{4}{3} & \frac{4}{3} & \frac{4}{3} & \frac{2}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & -\frac{1}{3} & 1 & 1 & 1 & \frac{5}{3} & 0 & 0 & 0 & \frac{2}{3} \\
\frac{2}{3} & \frac{2}{3} & \frac{2}{3} & 0 & -\frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} & -1 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 1 & -\frac{2}{3} & -\frac{2}{3} & -\frac{2}{3} & 0
\end{bmatrix}_{11}$$

$$\begin{bmatrix}
0 & 0 & 0 & -\frac{2}{3} & -1 & -1 & -1 & -\frac{5}{3} & -\frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} & \frac{1}{3} & -\frac{4}{3} & -\frac{4}{3} & -\frac{4}{3} & -\frac{2}{3} \\
0 & 0 & 0 & -\frac{2}{3} & -1 & -1 & -1 & -\frac{5}{3} & -\frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} & \frac{1}{3} & -\frac{4}{3} & -\frac{4}{3} & -\frac{4}{3} & -\frac{2}{3} \\
0 & 0 & 0 & -\frac{2}{3} & -1 & -1 & -1 & -\frac{5}{3} & -\frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} & \frac{1}{3} & -\frac{4}{3} & -\frac{4}{3} & -\frac{4}{3} & -\frac{2}{3} \\
\frac{2}{3} & \frac{2}{3} & \frac{2}{3} & 0 & -\frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} & -1 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 1 & -\frac{2}{3} & -\frac{2}{3} & -\frac{2}{3} & 0 \\
1 & 1 & 1 & \frac{1}{3} & 0 & 0 & 0 & -\frac{2}{3} & \frac{2}{3} & \frac{2}{3} & \frac{2}{3} & \frac{4}{3} & -\frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} & \frac{1}{3} \\
1 & 1 & 1 & \frac{1}{3} & 0 & 0 & 0 & -\frac{2}{3} & \frac{2}{3} & \frac{2}{3} & \frac{2}{3} & \frac{4}{3} & -\frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} & \frac{1}{3} \\
1 & 1 & 1 & \frac{1}{3} & 0 & 0 & 0 & -\frac{2}{3} & \frac{2}{3} & \frac{2}{3} & \frac{2}{3} & \frac{4}{3} & -\frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} & \frac{1}{3} \\
\frac{5}{3} & \frac{5}{3} & \frac{5}{3} & 1 & \frac{2}{3} & \frac{2}{3} & \frac{2}{3} & 0 & \frac{4}{3} & \frac{4}{3} & \frac{4}{3} & 2 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 1 \\
\frac{1}{3} & \frac{1}{3} & \frac{1}{3} & -\frac{1}{3} & -\frac{2}{3} & -\frac{2}{3} & -\frac{2}{3} & -\frac{4}{3} & 0 & 0 & 0 & \frac{2}{3} & -1 & -1 & -1 & -\frac{1}{3} \\
\frac{1}{3} & \frac{1}{3} & \frac{1}{3} & -\frac{1}{3} & -\frac{2}{3} & -\frac{2}{3} & -\frac{2}{3} & -\frac{4}{3} & 0 & 0 & 0 & \frac{2}{3} & -1 & -1 & -1 & -\frac{1}{3} \\
\frac{1}{3} & \frac{1}{3} & \frac{1}{3} & -\frac{1}{3} & -\frac{2}{3} & -\frac{2}{3} & -\frac{2}{3} & -\frac{4}{3} & 0 & 0 & 0 & \frac{2}{3} & -1 & -1 & -1 & -\frac{1}{3} \\
-\frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} & -1 & -\frac{4}{3} & -\frac{4}{3} & -\frac{4}{3} & -2 & -\frac{2}{3} & -\frac{2}{3} & -\frac{2}{3} & 0 & -\frac{5}{3} & -\frac{5}{3} & -\frac{5}{3} & -1 \\
\frac{4}{3} & \frac{4}{3} & \frac{4}{3} & \frac{2}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & -\frac{1}{3} & 1 & 1 & 1 & \frac{5}{3} & 0 & 0 & 0 & \frac{2}{3} \\
\frac{4}{3} & \frac{4}{3} & \frac{4}{3} & \frac{2}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & -\frac{1}{3} & 1 & 1 & 1 & \frac{5}{3} & 0 & 0 & 0 & \frac{2}{3} \\
\frac{4}{3} & \frac{4}{3} & \frac{4}{3} & \frac{2}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & -\frac{1}{3} & 1 & 1 & 1 & \frac{5}{3} & 0 & 0 & 0 & \frac{2}{3} \\
\frac{2}{3} & \frac{2}{3} & \frac{2}{3} & 0 & -\frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} & -1 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 1 & -\frac{2}{3} & -\frac{2}{3} & -\frac{2}{3} & 0
\end{bmatrix}_{21}$$

$$
\begin{bmatrix}
0 & 0 & 0 & -\frac{2}{3} & -1 & -1 & -1 & -\frac{5}{3} & -\frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} & \frac{1}{3} & -\frac{4}{3} & -\frac{4}{3} & -\frac{4}{3} & -\frac{2}{3} \\
0 & 0 & 0 & -\frac{2}{3} & -1 & -1 & -1 & -\frac{5}{3} & -\frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} & \frac{1}{3} & -\frac{4}{3} & -\frac{4}{3} & -\frac{4}{3} & -\frac{2}{3} \\
0 & 0 & 0 & -\frac{2}{3} & -1 & -1 & -1 & -\frac{5}{3} & -\frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} & \frac{1}{3} & -\frac{4}{3} & -\frac{4}{3} & -\frac{4}{3} & -\frac{2}{3} \\
\frac{2}{3} & \frac{2}{3} & \frac{2}{3} & 0 & -\frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} & -1 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 1 & -\frac{2}{3} & -\frac{2}{3} & -\frac{2}{3} & 0 \\
1 & 1 & 1 & \frac{1}{3} & 0 & 0 & 0 & -\frac{2}{3} & \frac{2}{3} & \frac{2}{3} & \frac{2}{3} & \frac{4}{3} & -\frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} & \frac{1}{3} \\
1 & 1 & 1 & \frac{1}{3} & 0 & 0 & 0 & -\frac{2}{3} & \frac{2}{3} & \frac{2}{3} & \frac{2}{3} & \frac{4}{3} & -\frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} & \frac{1}{3} \\
1 & 1 & 1 & \frac{1}{3} & 0 & 0 & 0 & -\frac{2}{3} & \frac{2}{3} & \frac{2}{3} & \frac{2}{3} & \frac{4}{3} & -\frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} & \frac{1}{3} \\
\frac{5}{3} & \frac{5}{3} & \frac{5}{3} & 1 & \frac{2}{3} & \frac{2}{3} & \frac{2}{3} & 0 & \frac{4}{3} & \frac{4}{3} & \frac{4}{3} & 2 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 1 \\
\frac{1}{3} & \frac{1}{3} & \frac{1}{3} & -\frac{1}{3} & -\frac{2}{3} & -\frac{2}{3} & -\frac{2}{3} & -\frac{4}{3} & 0 & 0 & 0 & \frac{2}{3} & -1 & -1 & -1 & -\frac{1}{3} \\
\frac{1}{3} & \frac{1}{3} & \frac{1}{3} & -\frac{1}{3} & -\frac{2}{3} & -\frac{2}{3} & -\frac{2}{3} & -\frac{4}{3} & 0 & 0 & 0 & \frac{2}{3} & -1 & -1 & -1 & -\frac{1}{3} \\
\frac{1}{3} & \frac{1}{3} & \frac{1}{3} & -\frac{1}{3} & -\frac{2}{3} & -\frac{2}{3} & -\frac{2}{3} & -\frac{4}{3} & 0 & 0 & 0 & \frac{2}{3} & -1 & -1 & -1 & -\frac{1}{3} \\
-\frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} & -1 & -\frac{4}{3} & -\frac{4}{3} & -\frac{4}{3} & -2 & -\frac{2}{3} & -\frac{2}{3} & -\frac{2}{3} & 0 & -\frac{5}{3} & -\frac{5}{3} & -\frac{5}{3} & -1 \\
\frac{4}{3} & \frac{4}{3} & \frac{4}{3} & \frac{2}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & -\frac{1}{3} & 1 & 1 & 1 & \frac{5}{3} & 0 & 0 & 0 & \frac{2}{3} \\
\frac{4}{3} & \frac{4}{3} & \frac{4}{3} & \frac{2}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & -\frac{1}{3} & 1 & 1 & 1 & \frac{5}{3} & 0 & 0 & 0 & \frac{2}{3} \\
\frac{4}{3} & \frac{4}{3} & \frac{4}{3} & \frac{2}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & -\frac{1}{3} & 1 & 1 & 1 & \frac{5}{3} & 0 & 0 & 0 & \frac{2}{3} \\
\frac{2}{3} & \frac{2}{3} & \frac{2}{3} & 0 & -\frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} & -1 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 1 & -\frac{2}{3} & -\frac{2}{3} & -\frac{2}{3} & 0
\end{bmatrix}_{12}
$$

$$
\begin{bmatrix}
0 & 0 & 0 & -\frac{2}{3} & -1 & -1 & -1 & -\frac{5}{3} & -\frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} & \frac{1}{3} & -\frac{4}{3} & -\frac{4}{3} & -\frac{4}{3} & -\frac{2}{3} \\
0 & 0 & 0 & -\frac{2}{3} & -1 & -1 & -1 & -\frac{5}{3} & -\frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} & \frac{1}{3} & -\frac{4}{3} & -\frac{4}{3} & -\frac{4}{3} & -\frac{2}{3} \\
0 & 0 & 0 & -\frac{2}{3} & -1 & -1 & -1 & -\frac{5}{3} & -\frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} & \frac{1}{3} & -\frac{4}{3} & -\frac{4}{3} & -\frac{4}{3} & -\frac{2}{3} \\
\frac{2}{3} & \frac{2}{3} & \frac{2}{3} & 0 & -\frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} & -1 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 1 & -\frac{2}{3} & -\frac{2}{3} & -\frac{2}{3} & 0 \\
1 & 1 & 1 & \frac{1}{3} & 0 & 0 & 0 & -\frac{2}{3} & \frac{2}{3} & \frac{2}{3} & \frac{2}{3} & \frac{4}{3} & -\frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} & \frac{1}{3} \\
1 & 1 & 1 & \frac{1}{3} & 0 & 0 & 0 & -\frac{2}{3} & \frac{2}{3} & \frac{2}{3} & \frac{2}{3} & \frac{4}{3} & -\frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} & \frac{1}{3} \\
1 & 1 & 1 & \frac{1}{3} & 0 & 0 & 0 & -\frac{2}{3} & \frac{2}{3} & \frac{2}{3} & \frac{2}{3} & \frac{4}{3} & -\frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} & \frac{1}{3} \\
\frac{5}{3} & \frac{5}{3} & \frac{5}{3} & 1 & \frac{2}{3} & \frac{2}{3} & \frac{2}{3} & 0 & \frac{4}{3} & \frac{4}{3} & \frac{4}{3} & 2 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 1 \\
\frac{1}{3} & \frac{1}{3} & \frac{1}{3} & -\frac{1}{3} & -\frac{2}{3} & -\frac{2}{3} & -\frac{2}{3} & -\frac{4}{3} & 0 & 0 & 0 & \frac{2}{3} & -1 & -1 & -1 & -\frac{1}{3} \\
\frac{1}{3} & \frac{1}{3} & \frac{1}{3} & -\frac{1}{3} & -\frac{2}{3} & -\frac{2}{3} & -\frac{2}{3} & -\frac{4}{3} & 0 & 0 & 0 & \frac{2}{3} & -1 & -1 & -1 & -\frac{1}{3} \\
\frac{1}{3} & \frac{1}{3} & \frac{1}{3} & -\frac{1}{3} & -\frac{2}{3} & -\frac{2}{3} & -\frac{2}{3} & -\frac{4}{3} & 0 & 0 & 0 & \frac{2}{3} & -1 & -1 & -1 & -\frac{1}{3} \\
-\frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} & -1 & -\frac{4}{3} & -\frac{4}{3} & -\frac{4}{3} & -2 & -\frac{2}{3} & -\frac{2}{3} & -\frac{2}{3} & 0 & -\frac{5}{3} & -\frac{5}{3} & -\frac{5}{3} & -1 \\
\frac{4}{3} & \frac{4}{3} & \frac{4}{3} & \frac{2}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & -\frac{1}{3} & 1 & 1 & 1 & \frac{5}{3} & 0 & 0 & 0 & \frac{2}{3} \\
\frac{4}{3} & \frac{4}{3} & \frac{4}{3} & \frac{2}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & -\frac{1}{3} & 1 & 1 & 1 & \frac{5}{3} & 0 & 0 & 0 & \frac{2}{3} \\
\frac{4}{3} & \frac{4}{3} & \frac{4}{3} & \frac{2}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & -\frac{1}{3} & 1 & 1 & 1 & \frac{5}{3} & 0 & 0 & 0 & \frac{2}{3} \\
\frac{2}{3} & \frac{2}{3} & \frac{2}{3} & 0 & -\frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} & -1 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 1 & -\frac{2}{3} & -\frac{2}{3} & -\frac{2}{3} & 0
\end{bmatrix}_{22}
$$

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